Research Statement

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My research interests lie in the theoretical and numerical analysis of nonlinear partial differential equations. In particular, my current research focuses on fluid mechanics, kinetic theory, and mathematical models arising from physics and biology. These models, combining *nonlinearity* and *nonlocality*, are challenging for both analysis and computations. The primary goal is to design and analyze PDE models, develop a suite of flexible and novel mathematical tools to understand the nonlocal and nonlinear structures of the equations, and capture the underlying physical and biological phenomena, as well as mathematical properties. Also, design efficient and reliable numerical algorithms to solve the equations and verify the analytical results.

One subject I have been working on is the *mathematical models of self-organized dynamics.* Collective behaviors are commonly observed in nature and human societies. There has been growing interest in the development of the mathematical theory of complex physical and biological systems. In particular, I am working on models of self-organized dynamics, investigating the emergence of small-scale interactions towards global phenomena, such as flocking, swarming and clustering. A multi-scale framework originally developed for gas dynamics can be adapted to these biological systems. Powerful tools in kinetic theory and fluid mechanics are being applied and further developed to understand the nonlinear and nonlocal structure of the systems, as well as the connections to real-world applications. I am also working on some numerical aspects of kinetic equations, taking into account the nonlocal feature of these biological models.

Another subject I am interested in is the *regularity and singularity formations in Eulerian dynamics.* The Euler equations in fluid mechanics describe the motion of fluids like water and air. Together with Navier-Stokes equation, they are considered as the most fundamental systems in fluid dynamics. Though the equations have been proposed and studied for over two hundred years, some key mathematical features are still not well understood and remain to be the most challenging problems in fluid mechanics. One major open problem on the 3D Euler equations is whether solutions are globally regular, or there could be finite time singularity formation. I am working on 1D and 2D related models which share some structures of 3D Euler equations. These models have physical and biological meanings of their own. Meanwhile, they could provide insight into the possible regularity or singularity formations of 3D Euler equation. New techniques are under development, which aim to provide a universal tool to analyze these equations and systems in fluid mechanics.

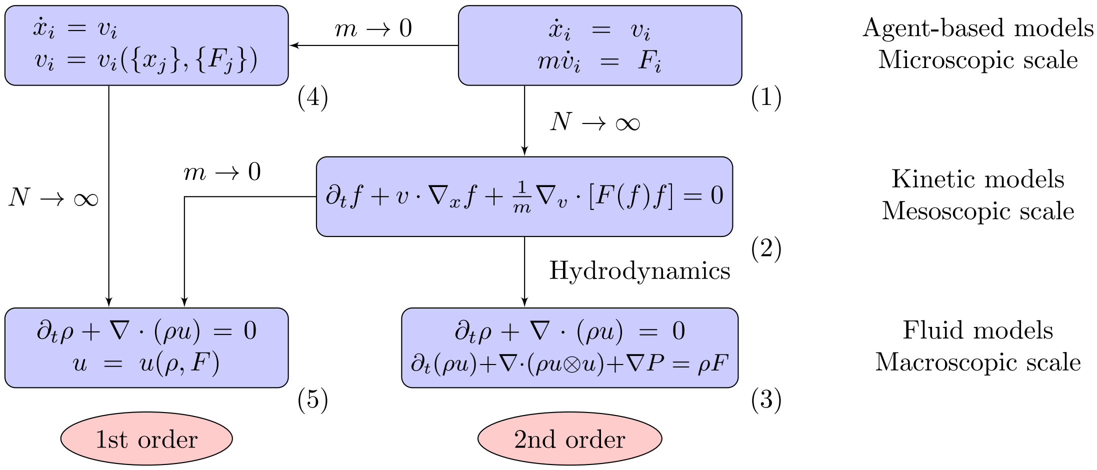
The rest of the statement includes the outline of some works I am involved in encompassing these two main subjects. Other related topics are briefly discussed in the last section.

1 Mathematical theory for self-organized dynamics

Collective behaviors are widely manifested in many biological contexts: schools of fish, flocks of birds, colonies of ants, etc. Such complex self-organized systems have attracted a lot of attention to scientists, including mathematicians. Over the last decade, a lot of effort has been made to develop a mathematical framework to understand the emergence of complex behavior in biological and social systems.

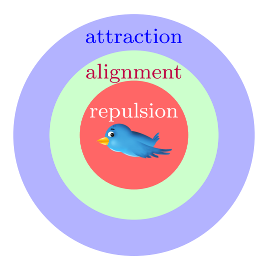
1.1 A multiscale framework on swarming dynamics

A celebrated multiscale framework is being constructed to model self-organized systems. It follows the recipe of Boltzmann's kinetic theory, which has been very successful in modeling gas dynamics and also fluid dynamics as suitable hydrodynamic limits.



The diagram above illustrates the framework, starting from a second order agent-based model on Newton's second law (1), where small-scale interactions are modeled through interaction forces F_i. When the number of agents N becomes large, a kinetic equation (2) can be derived which serves as a mean-field limit of the microscopic dynamics. Furthermore, taking moments on (2) will yield the macroscopic descriptions of the system (3): Eulerian dynamics with interaction forces. Both the kinetic and fluid representations of the dynamics are of great interest to mathematicians, physicists, and biologists.

I have been working on the mathematical theory of kinetic and fluid models for self-organized dynamics. In particular, the interaction forces are modeled through a *three-zone interaction* framework [24], illustrated in the right graph, including long-range attraction, mid-range alignment and short-range repulsion.



The alignment interaction describes the mechanism where individuals align their velocity with neighbors. This leads to global *flocking* phenomena, which is widely observed in animal swarms. One celebrated agent-based flocking models is proposed by Cucker-Smale [17], with the interaction forces

$$F_i = \frac{1}{N}\sum_{j=1}^{N}\phi(|x_i - x_j|)(v_j - v_i), \tag{6}$$

where the *influence function* ϕ measures the strength of interactions between two agents. Naturally, it is a decreasing function with respect to the distance between agents. Despite of the complex coupling, Cucker-Smale model enjoys the flocking property. Other flocking dynamics include Vicsek model [25], Motsch-Tadmor model [22], and more.

The attraction-repulsion interaction is usually modeled through an interaction potential. Such force has been extensively studied in the context of aggregation equations.

1.2 Kinetic swarming models

The kinetic swarming models (2) has been investigated in the recent decade, with different choices of nonlocal interaction forces.

The kinetic flocking models with Cucker-Smale interaction force (6) has the form

$$\partial_t f + v \cdot \nabla_x f = \frac{1}{m} \nabla_v \cdot \left(\iint \phi(|x-y|)(v-v^*) f(x,v) f(y,v^*) dy dv^* \right). \tag{7}$$

Global wellposedness has been proved in [14]. More importantly, it is shown that the flocking phenomenon persists in the kinetic level. In the work [11], I give an alternative proof of the flocking property of (7), and extend the result to the more delicate Motsch-Tadmor model.

The flocking behavior implies that the solution concentrates in velocity variable as time approaches infinity, namely $f(t,x,v) \to \rho(t,x)\delta_{v=\bar{v}}$ as $t \to \infty$. Such type of asymptotic δ-singularity brings challenges to the numerical implementation, as the classical approximation methods suffer instability due to Gibbs phenomenon. In [11], I design a scheme based on discontinuous Galerkin method, and prove accuracy and stability of the scheme. The method can be easily extended to Motsch-Tadmor alignment force, as well as the inclusion of attraction-repulsion forces.

With Rey in [8], we propose a new scheme based on *velocity scaling method*. The key idea is to express the solution $f(t,x,v) = \omega^n(t,x) g(t,x,\omega(t,x)(v-u(t,x)))$, where ω is a dynamic scaling factor on velocity variable, and u is the dynamical average of the velocity. We find an exact scaling factor so that the rescaled function g does not approach a singular asymptotic profile. Appropriate numerical schemes are designed for the dynamics of (g,u,ω), without worrying about the asymptotic δ-singularity. The velocity scaling method has the potential to be applied to more general kinetic equations, some of which are under investigation now.

The kinetic models (2) with potential forces also attract a lot of attention. In particular, with repulsive Newtonian force, it is known as *Vlasov-Poisson equation* in plasma physics. Global regularity is a long-standing problem, and has been recently proved by [23]. For less singular potential, global wellposedness is easier to obtain. See for instance [5] for global regularity on kinetic swarming model general three-zone interaction forces.

1.3 Euler-Alignment system

The macroscopic flocking model (3) with Cucker-Smale interaction force (6) is called *Euler-Alignment system*. For the pressureless case $P \equiv 0$, the system can be written as

$$\partial_t \rho + \nabla \cdot (\rho u) = 0, \qquad \partial_t u + u \cdot \nabla u = \int_{\mathbb{R}^n} \phi(|x-y|)(u(t,y) - u(t,x)) \rho(t,y) dy. \tag{8}$$

Without the alignment force ($\phi \equiv 0$), the system is the pressureless compressible Euler equation. It is well-known that the nonlinear convection term causes a finite time formation of *singular shock*. On the other hand, the alignment force intends to regularize the equation. The understanding of the competition between the two effects is important and challenging, especially because of the nonlinear and nonlocal structures.

1.3.1 Bounded Lipschitz influence function

A biologically relevant assumption on the influence function ϕ is that it is a bounded Lipschitz decreasing function. This setup has been studied in a joint work with Tadmor [10]. We show a *critical threshold phenomenon*: subcritical initial data leads to global smooth solutions, while supercritical initial data leads to finite time blowup. The result works for both 1D and 2D Euler-Alignment system, as well as the system where Cucker-Smale alignment force is replaced by Motsch-Tadmor alignment force. Moreover, we show flocking phenomenon for these macroscopic systems.

In a successive work with Carrillo, Choi and Tadmor [1], we obtain a sharp critical threshold for (8) in 1D: the system has a global strong solution if and only if the initial data (ρ_0, u_0) satisfy $\partial_x u_0 + \phi * \rho_0 \geq 0$ for all $x \in \mathbb{R}$. This is the first sharp critical threshold result for a system with nonlocal forces. General systems with three-zone interaction forces are also discussed, including the difficult case when the attraction-repulsion force is singular.

1.3.2 Singular influence function

Another interesting type of influence functions has the form $\phi(r) = c_\alpha r^{-(n+\alpha)}$, which is singular at $r = 0$. System (8) with such influence function is very similar to *fractional Burgers equation*

$$\partial_t u + u \cdot \nabla u = -(-\Delta)^{\alpha/2} u = c_\alpha \int_{\mathbb{R}^n} \frac{u(t,y) - u(t,x)}{|x-y|^{n+\alpha}} dy. \tag{9}$$

Equation (9) has been studied in [18]. The result says that if $\alpha < 1$, then there exists initial data which lead to finite time blowup; if $\alpha \geq 1$, then all smooth initial data lead to global regularity.

Though the only difference is that the alignment integral in (8) has a weight $\rho(t, \cdot)$, the global behavior of (8) is very different from (9). With Do, Kiselev and Ryzhik in [4], we prove global regularity of (8) with singular influence function in 1D periodic setup, for all smooth initial data and $\alpha > 0$, as long as the initial data $\rho_0 > 0$ is strictly positive. In particular, the behavior of solution is completely different from (9) when $0 < \alpha < 1$. A new mechanism is discovered: regularization due to the nonlocal nonlinear modulation of dissipation. The proof is highly non-trivial, where new techniques are developed to treat the nonlinear nonlocal dissipation, using the method on the modulus of continuity [19] as well as nonlinear maximum principle [16].

With Kiselev in [7], we generalize the global regularity result for a large class of singular alignment forces. Moreover, we consider the three-zone model with singular attraction-repulsion forces. We show that the singular alignment produces strong dissipation which dominates the concentration effect of the attraction, even if the attractive force is very singular.

The extension of the results to higher dimensional systems are also challenging and currently under investigation.

1.4 Zero inertia limit and first order systems

There is another special type of macroscopic swarming models that I am interested in. It is called first order system (5). The system can be derived from the kinetic swarming model (2) by taking the total mass $m \to 0$. It corresponds to the *zero inertia limit*. Alternatively, one can start with the agent-based model (1) and take the limit $m \to 0$ first. This yields a first order agent-based model (4). Then, (5) can be considered as the mean-field limit of (4).

One classical first order system is the *aggregation equation*, where $u = -\nabla K * \rho$. It has been extensively studied in the last two decades by a variety of mathematicians. It fits into the multi-scale framework with $F_i = -v_i + \sum_j \nabla_{x_i} K(x_i - x_j)$. While the second part represents attraction-repulsion, the first term $-v_i$ does not have a good biological meaning.

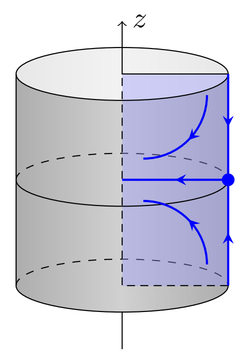
In a joint work with Fetecau and Sun [5], we derive a first order system for the three-zone model, where the $-v_i$ term is replaced by Cucker-Smale alignment force (6). The macroscopic system is a biologically motivated aggregation equation with alignment. We studied the global wellposedness of the system, as well as the rigorous passage of limit from the kinetic swarming model (2). Highly non-trivial techniques are introduced to treat the nonlocal alignment operator. I am working on the asymptotic behavior of (5), which is believed to be similar as aggregation equation.

The zero inertia limit is mono-kinetic, namely, it has the form $\rho(t,x)\delta_{v=u(t,x)}$, where (ρ, u) satisfy the macroscopic system. Such singular limit brings difficulties in designing numerical schemes which are compatible for (2) with any m, as well as the limiting system (5) as $m \to 0$. Such schemes are called *asymptotic-preserving schemes*. With Chertock and Yan in [3], we design an asymptotic-preserving scheme for the zero inertia limit, using velocity scaling method to bypass the difficulty from the singular limit. The idea can be extended to other kinetic systems with singular limits.

2 Singularity formation in Euler equation and related models

Understanding the 3D incompressible Euler equation is one of the most challenging problems in modern applied analysis. One major question is whether solutions corresponding to smooth initial data remain globally regular, or could blow up in finite time.

Recently, Hou and Luo in [21] proposed a new scenario for potential finite time blowup in the 3D Euler equation, based on extensive numerical simulation. As illustrated in the right graph, they consider axisymmetric data in a cylindrical domain. The blowup is observed near the hyperbolic point at the boundary of the cylinder. Such blowup scenario motivates a major work of Kiselev and Šverák [20] on the sharp fast growth of solutions for the 2D Euler equation, where there is no vorticity stretching.



To understand the complex behavior of the solution for 3D Euler equation, many 1D and 2D simplified models are proposed and analyzed. These models, some of which are interesting by themselves, provide insights to the possible scenarios of either blowup or regularity.

2.1 Hyperbolic Boussinesq system

If we take a cross-section (blue plane in the graph) of the Hou-Lou scenario, the dynamics is governed by the *2D inviscid Boussinesq equation*. The Boussinesq equations model large scale atmospheric and oceanic flows that are responsible for cold fronts and the jet stream. Whether the inviscid Boussinesq equation has global regularity or not is a hard open problem by its own.

Several 1D simplified models have been proposed, and a finite time blowup behavior is observed. I am interested in the singularity formation of 2D Boussinesq-like equations.

With Kiselev in [6], we study a 2D hyperbolic Boussinesq system, which shares similar structures as the Boussinesq system. The model has an incompressible flow and a slightly different stretching term modeling buoyancy. We prove that finite time blowup happens for a natural class of initial data. It is the first blowup result for 2D incompressible Boussinesq-like systems. We are currently trying to bridge the gaps between the hyperbolic system and the real 2D Boussinesq system, and the ultimate goal is to understand the blowup behavior of the 2D Boussinesq equation analytically.

2.2 Fractional porous medium equation

Another 2D model which inherits the structure of 3D Euler equations is the *surface quasi-geostrophic (SQG) equation*. The relation is first discovered by Constantin, Majda and Tabak in [15]. However, The global behavior of 2D SQG equation remains to be an open problem.

Chae, Córdoba, Córdoba and Fontelos in [2] consider a 1D simplified model of 2D SQG equation

$$\partial_t \rho + \partial_x(\rho u) = 0, \quad u = H\Lambda^{\alpha-1}\rho. \tag{10}$$

with $\alpha = 1$. They prove global regularity for $\rho_0 > 0$, and construct an initial data $\rho_0 \geq 0$ which leads to a finite time blowup.

The general system (10) with $\alpha \in (0, 2)$ can be viewed as porous medium equation with a nonlocal fractional potential pressure. Caffarelli and Vázquez study the system in [13]. They establish a global wellposedness theory for L^1 data, and prove that any L^1 initial data are regularized to C^γ instantaneously, for some $\gamma > 0$.

In [4], we observe that (10) is a special case of the Euler-Alignment system (8) with fractional influence function. We prove global regularity for $\rho_0 > 0$. On the other hand, I construct a family of $\rho_0 \geq 0$ in [12] and prove that the solution loses C^α regularity in finite time. These results non-trivially extend the result in [2] to more general power α. Moreover, the blowup result shows that the regularization effect in [13] cannot be strengthened to C^α.

I am also working on other 1D or 2D models, where the behaviors of the solutions could ultimately provide hints to the understanding of the 3D Euler equation.

3 More topics and future trends

There are many related topics in fluid dynamics and kinetic theory that I have been working on. They are motivated by physical and biological applications, and they share interesting nonlinearity and nonlocality with the two main subjects. The analytical and numerical tools developed for fluid equations and self-organized dynamics can be applied to these challenging problems. I will list these problems without getting into details.

- Mixing by incompressible flows.
- Keller-Segel equation on chemotaxis, and coupling with fluid equations.
- Coagulation-fragmentation models and gelation formations.
- Spectral gap in 2D Eulerian dynamics.

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