MATH 524 Lecture Notes

Kinetic theory and connections to fluid mechanics

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Contents

Chapter 1

Introduction to Boltzmann equation

1.1 A brief history on gas dynamics

1.1.1 Before Boltzmann equation

- Boyle's law (1650): $PV =$ constant, where P is the pressure, V is the volume.
- Bernoulli (1738): $PV \propto c^2$, c is the speed. c^2 is viewed as T temperature.
- Clausius (1857): particle interaction. Introduce *mean free path*.
- Maxwell (1860, 1866): Maxwell equation, put together Newtonian dynamics, fluid dynamics and the concept of energy.
- Boltzmann (1872): Boltzmann equation.

1.1.2 Length scales

Let N be number of particles in $\Omega \subset \mathbb{R}^D$, where $D(=3)$ is the dimension. Let R be the interaction range. For hard sphere collision, $R = 2a$, where a is the radius of the molecule.

There are three different length scales:

- Macroscopic length scale: $L = V^{\frac{1}{D}}$,
- Intermolecular spacing: $\Lambda = \left(\frac{V}{N}\right)^{\frac{1}{D}}$,
- Range of interaction: R.

The *Ideal gas* regime:

$$R \ll \Lambda \ll L, \qquad \text{i.e.} \quad NR^D \ll V \quad \text{and} \quad 1 \ll N.$$

If R is closer to Λ, we get liquid or even solid.

A fourth length scale introduced by Clausius is the *mean free path*, denoted by λ. It characterizes distance molecule travels between collisions. We have the following identity.

$$Na^{D-1}\lambda|\mathbb{B}^{D-1}| + Na^D|\mathbb{B}^D| = V,$$

where $\mathbb{B}^D$ is the unit ball in dimension D. For ideal gas, we have

$$\lambda \sim \frac{V}{NR^{D-1}}.$$

Exercise 1. *Check that for ideal gas, $\lambda \gg \Lambda$.*

λ could be bigger or smaller to the macroscopic length scale L. Their ratio λ/L is called *Knudsen number*:

$$\text{Kn} = \frac{\lambda}{L} = \frac{V^{\frac{D-1}{D}}}{NR^{D-1}}.$$

The Knudsen number $\text{Kn} = \lambda/L$ identifies three regimes:

- $\text{Kn} \ll 1$: collisional **fluid** regime,
- $\text{Kn} \sim 1$: collisional kinetic regime, $NR^{D-1} = \mathcal{O}(L^{D-1})$, known as *Boltzmann-Grad scaling*,
- $\text{Kn} \gg 1$: **collisionless** kinetic regime.

1.1.3 Maxwell kinetic equation

The *kinetic density* is a distribution function in the particle phase space, denoted by

$$f \equiv f(t, x, v), \quad t > 0, x \in \Omega, v \in \mathbb{R}^D.$$

The dynamics of the kinetic density is given as

$$\partial_t f + v \cdot \nabla_x f = \frac{1}{\text{Kn}}\mathcal{B}[f, f],$$

including free transport and collision. The Knudsen number enters naturally by doing appropriate scaling.

In the collisionless limit when $\text{Kn} \to \infty$, we have a pure transport equation:

$$\partial_t f + v \cdot \nabla_x f = 0, \quad \Rightarrow \quad \text{the solution is } f(t, x, v) = f^{in}(x - vt, v).$$

In the fluid limit when $\text{Kn} \to 0$, we get

$$\mathcal{B}[f, f] = 0, \quad \Rightarrow \quad \text{we call the solution } \textit{Maxwellian.}$$

1.2 From Newtonian to Boltzmann

1.2.1 Elastic binary collisions

The collision integral $\mathcal{B}$ satisfies the following conditions, under Boltzmann-Grad scaling:

- $R \to 0$, so collision happens when two particles are at the same position. So, $\mathcal{B}$ is spatially homogeneous.
- The gas is dilute enough so that we only count *binary collisions*.
- Collisions are *elastic*, namely energy is conserved during collision.

Let v_1', v_2' be the pre-collision velocity of the two particles, and v_1, v_2 be the post-collision velocity. The collision should satisfy the following conservation laws:

- Conservation of mass: $m_1 + m_2 = m_1 + m_2$,
- Conservation of momentum: $m_1 v_1' + m_2 v_2' = m_1 v_1 + m_2 v_2$,
- Conservation of energy: $m_1 |v_1'|^2 + m_2 |v_2'|^2 = m_1 |v_1|^2 + m_2 |v_2|^2$.

Use the *center of mass coordinate system*

$$M = m_1 + m_2, \quad m = \frac{m_1 m_2}{M}, \quad V = \frac{m_1 v_1 + m_2 v_2}{M}, \quad v = v_2 - v_1.$$

Then we have

$$v_1 = V - \frac{m}{m_1} v, \quad v_2 = V + \frac{m}{m_2} v, \quad m_1 |v_1|^2 + m_2 |v_2|^2 = M|V|^2 + m|v|^2,$$

and the constraints become

$$MV' = MV, \quad M|V'|^2 + m|v'|^2 = M|V|^2 + m|v|^2, \quad \Rightarrow \quad V' = V, \quad |v'| = |v|.$$

Given m_1, m_2, v_1, v_2, we can express v_1', v_2' as

$$v_1' = V - \frac{m}{m_1} |v| \sigma, \quad v_2' = V + \frac{m}{m_2} |v| \sigma,$$

where the *deviation parameter* $\sigma \in \mathbb{S}^{D-1}$ contains $D-1$ degree of freedom, which makes sence as there are $2D$ unknowns and $D+1$ constraints. Note that σ represents the angle between v_1' and v_2'.

In particular, if $m_1 = m_2$, then we have

$$v_1' = V - \frac{|v|}{2} \sigma, \quad v_2' = V + \frac{|v|}{2} \sigma.$$

An alternative parametrization is to use the *deflection parameter* $\omega \in \mathbb{S}^{D-1}$, and

$$v_1' = v_1 - \omega \cdot (v_1 - v_2) \omega, \quad v_2' = v_2 + \omega \cdot (v_1 - v_2) \omega,$$

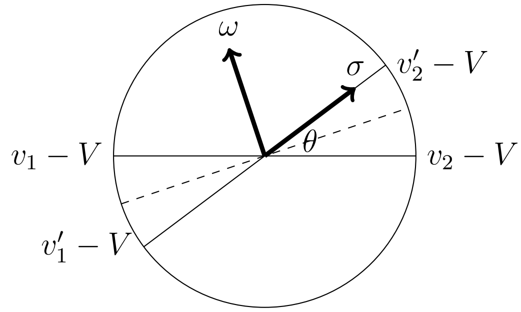

Figure 1.1: Illustration of the free parameters σ and ω

One can find ω by normalizing $\frac{v_1 - v_2}{|v_1 - v_2|} + \sigma$. It is also easy to see that

$$\omega = \frac{v_1 - v_1'}{|v_1 - v_1'|} = \frac{v_2' - v_2}{|v_2 - v_2'|}.$$

Figure 1.1 illustrates the geographic meanings of σ and ω.

The free parameters describe the *scattering* phenomenon. Indeed, for a fixed $R > 0$, if the collision happens at (x_1, x_2) where $|x_1 - x_2| = R$, then ω satisfies

$$\omega = \frac{x_1 - x_2}{|x_1 - x_2|}.$$

Since the collision is elastic, it is *reversible*. However, one can distinguish the pre-collision configuration and post-collision configuration as follows:

$$(v_1' - v_2') \cdot \omega < 0, \quad (v_1 - v_2) \cdot \omega > 0.$$

One special case is when $(v_1' - v_2') \cdot \omega = 0$. It represents two particles touch and travel with the same velocity. This is called *grazing collision*, which can be neglected as it is a rare event.

Exercise 2. *Check the following identity:*

$$(v_1 - v_2) \cdot \omega + (v_1' - v_2') \cdot \omega = 0. \tag{1.1}$$

1.2.2 Derivation of Boltzmann equation: hard sphere

Let us consider the case when all particles have same mass m. For N particles at positions $(x_1, \cdots, x_N)$ and velocities $(v_1, \cdots, v_N)$. Denote

$$\mathcal{D}_s := \{\mathbf{x} \in \mathbb{R}^{Ds} \mid \forall 1 \le i \ne j \le s, |x_i - x_j| > R\}, \quad s = 1, \cdots, N.$$

It represents the spatial configurations such that there is no collisions. The Newtonian flow reads

$$\frac{dx_i}{dt} = v_i, \quad \frac{dv_i}{dt} = 0, \quad \text{on } \mathcal{D}_N, \quad \forall\ 1 \le i \le N,$$
$$v_i' = v_i - \omega_{ij} \cdot (v_i - v_j)\omega_{ij}, \quad v_j' = v_j + \omega_{ij} \cdot (v_1 - v_2)\omega_{ij}, \quad \text{if } \exists j \ne i, |x_i - x_j| = R.$$

The Liouville equation related to the particle system is

$$\partial_t f_N + \sum_{i=1}^{N} v_i \cdot \nabla_{x_i} f_N = 0, \quad \text{in } \mathbb{R}^+ \times \mathcal{D}_N \times \mathbb{R}^{DN}, \tag{1.2}$$

with boundary condition $f_N(t, \mathbf{x}, \mathbf{v}') = f_N(t, \mathbf{x}, \mathbf{v})$. As particles are indistinguishable, f_N is invariant under permutation. The system is too big and not practical when N is big. One needs to simplfy the dynamics. One way to do it is to use the *BBGKY hierarchy.*

Take the s-th marginal of the distribution f_N,

$$f_N^{(s)}(t, \underbrace{x_1, v_1, \cdots, x_s, v_s}_{:=Z_s}) = \int_{\mathbb{R}^{2D(N-s)}} f_N(t, \mathbf{x}, \mathbf{v}) \mathbb{1}_{\mathbf{x} \in \mathcal{D}_N} \underbrace{dx_{s+1} dv_{s+1} \cdots dx_n dv_n}_{:=dZ_s^-}. \tag{1.3}$$

Let us integrate (1.2) in $\mathbb{1}_{\mathbf{x} \in \mathcal{D}_N} dZ_s^-$ and get the dynamics of the marginal distributions:

$$\partial_t f_N^{(s)} + \sum_{i=1}^{N} \int_{\mathbb{R}^{2D(N-s)}} v_i \cdot \nabla_{x_i} f_N \mathbb{1}_{\mathbf{x} \in \mathcal{D}_N} dZ_s^- = 0.$$

Take a weak formulation by testing on $\phi(Z_s)$ for $Z_s \in \mathcal{D}_s \times \mathbb{R}^D$.

$$\partial_t \int_{\mathbb{R}^{2Ds}} f_N^{(s)}(t, Z_s)\phi(Z_s) dZ_s = -\sum_{i=1}^{N} \int_{\mathcal{D}_N \times \mathbb{R}^{DN}} v_i \cdot \nabla_{x_i} f_N \phi(Z_s) dZ_N =: \text{RHS}.$$

For the right hand side, we can do integration by parts and get

$$\text{RHS} = \sum_{i=1}^{s} \int_{\mathcal{D}_N \times \mathbb{R}^{DN}} v_i \cdot \nabla_{x_i} \phi(Z_s) f_N dZ_N + \sum_{i=1}^{N} \sum_{j=1}^{N} I_{ij},$$

where

$$I_{ij} = -R^{D-1} \int_{\mathbb{S}^{D-1} \times \mathbb{R}^D \times \mathbb{R}^{2D(N-1)}} v_i \cdot n_{ij} f_N(Z_{i-1}, x_j + R\omega, v_i, Z_i^-)\phi(Z_s) d\omega dv_i dZ_{i-1} dZ_i^-.$$

Here, n_{ij} is the outer normal direction of the boundary of $|x_i - x_j| > R$, which is the inner normal direction of the ball $B_R(x_j)$ at point x_i, namely

$$n_{ij} = \frac{x_j - x_i}{|x_i - x_j|} = -\omega_{ij} =: -\omega.$$

Exercise 3. *Use the boundary condition and* (1.1) *to prove*

$$\sum_{i=1}^{s}\sum_{j=1}^{s} I_{ij} = 0, \quad \sum_{i=s+1}^{N}\sum_{j=s+1}^{N} I_{ij} = 0.$$

Due to invariance under permutation, we can simplify $\sum_{i=1}^{N}\sum_{j=1}^{N} I_{ij}$ as follows:

$$\sum_{i=1}^{N}\sum_{j=1}^{N} I_{ij} = \sum_{i=1}^{s}\sum_{j=s+1}^{N} I_{ij} + \sum_{i=s+1}^{N}\sum_{j=1}^{s} I_{ij} = (N-s)\sum_{i=1}^{s}(I_{i,s+1} + I_{s+1,i})$$
$$= (N-s)R^{D-1}\sum_{i=1}^{s}\int_{\mathbb{S}^{D-1}\times\mathbb{R}^D\times\mathbb{R}^{2Ds}} (v_{s+1}-v_i)\cdot\omega f_N^{(s+1)}(Z_s, x_j + R\omega, v_{s+1})\phi(Z_s)d\omega dv_{s+1}dZ_s.$$

Put everything together, we get the dynamics of $f_N^{(s)}$, in the strong formulation as

$$\partial_t f_N^{(s)} + \sum_{i=1}^{s} v_i\cdot\nabla_{x_i} f_N^{(s)} = \mathcal{C}_{s,s+1}[f_N^{(s+1)}], \quad \text{in } \mathbb{R}_+\times\mathcal{D}_s\times\mathbb{R}^{DN},$$

with the collision kernel $\mathcal{C}_{s,s+1}$ as

$$\mathcal{C}_{s,s+1}[f_N^{(s+1)}](t,Z_s) := (N-s)R^{D-1}\sum_{i=1}^{s}\int_{\mathbb{S}^{D-1}\times\mathbb{R}^D}(v_{s+1}-v_i)\cdot\omega f_N^{(s+1)}(t,Z_s,x_i+R\omega,v_{s+1})d\omega dv_{s+1}.$$

In particular, for $s=1$, the dynamics of the first marginal reads

$$\partial_t f_N^{(1)} + v_1\cdot\nabla_{x_1} f_N^{(1)} = (N-1)R^{D-1}\int_{\mathbb{S}^{D-1}\times\mathbb{R}^D}(v_2-v_1)\cdot\omega f_N^{(2)}(t,x_1,v_1,x_1+R\omega,v_2)d\omega dv_2,$$

for $(t,x_1,v_1)\in\mathbb{R}_+\times\mathcal{D}_1\times\mathbb{R}^D$, with boundary condition

$$f(t,x_1,v_1',x_2,v_2') = f(t,x_1,v_1,x_2,v_2).$$

Now, we can formally take the Boltzmann-Grad limit: $R\to 0, N\to\infty$ and $(N-1)R^{D-1}$ converges to a constant C. We get

$$\partial_t f_\infty^{(1)} + v_1\cdot\nabla_{x_1} f_\infty^{(1)} = C\int_{\mathbb{S}^{D-1}\times\mathbb{R}^D}(v_2-v_1)\cdot\omega f_\infty^{(2)}(t,x_1,v_1,x_1,v_2)d\omega dv_2.$$

The boundary condition becomes spatially homogeneous,

$$f(t,x_1,v_1',x_2,v_2') = f(t,x_1,v_1,x_1,v_2), \quad \forall\ x_1\in\Omega.$$

Next, we impose a very important *chaotic assumption*: collisions involve only uncorrelated particles, namely

$$f_\infty^{(2)}(t,x_1,v_1,x_2,v_2) = f_\infty^{(1)}(t,x_1,v_1)\cdot f_\infty^{(1)}(t,x_2,v_2).$$

The assumption is not true for N-particle system as previous collisions can happen before the current collision, and particles are not independent. However, when $N \to \infty$, Lanford gave a rigorous proof in 1978 that such chaotic assumption holds.

Let us switch our notations to a more convenient set: $x_1 \to x, v_1 \to v, v_2 \to v_*$ and $f_\infty^{(1)} \to f$. Then, we can write

$$\partial_t f + v \cdot \nabla_x f = C \int_{\mathbb{S}^{D-1}\times\mathbb{R}^D} (v_* - v) \cdot \omega f(t,x,v) f(t,x,v_*) d\omega dv_*.$$

Finally, we decompose the collision kernel into two terms by seperating the pre- and post-collision configurations

$$(v_* - v) \cdot \omega = \underbrace{[(v_* - v) \cdot \omega]_+}_{\text{Post-collision}} - \underbrace{[(v_* - v) \cdot \omega]_-}_{\text{Pre-collision}},$$

representing the gain term and the loss term, respectively.

For the loss term, applying boundary condition and (1.1), we get

$$\begin{aligned}&\int_{\mathbb{S}^{D-1}\times\mathbb{R}^D} [(v_* - v) \cdot \omega]_- \, f(t,x,v) f(t,x,v_*) d\omega dv_* \\ &\qquad = \int_{\mathbb{S}^{D-1}\times\mathbb{R}^D} [(v_*' - v') \cdot \omega]_+ \, f(t,x,v') f(t,x,v_*') d\omega dv_*.\end{aligned}$$

We end up with Boltzmann equation with hard sphere collision operator

$$\mathcal{B}[f,f](t,x,v) = \int_{\mathbb{S}^{D-1}\times\mathbb{R}^D} \underbrace{C\,[(v - v_*) \cdot \omega]_+}_{:=b(|v_*-v|,\theta)} \underbrace{\big(f(t,x,v')f(t,x,v_*') - f(t,x,v)f(t,x,v_*)\big)}_{:=f'f_*'-ff_*} d\omega dv_*.$$

In general, the collision operator can be written as

$$\mathcal{B}[f,f](t,x,v) = \int_{\mathbb{S}^{D-1}\times\mathbb{R}^D} b(v_* - v, \omega)(f'f_*' - ff_*) d\omega dv_*, \tag{1.4}$$

where b is the *collisional cross-section* which only depends on the relative speed $|v_* - v|$ and the deflection angle β (the angle between $v_* - v$ and ω). For hard sphere case,

$$b(v_* - v, \omega) = C|(v_* - v) \cdot \omega| = C|v_* - v||\cos\beta|.$$

1.2.3 Derivation of Boltzmann equation: soft potential

More general types of collisions can be modeled through a potential Φ_R as the following Hamiltonian dynamics:

$$\frac{d}{dt}x_i = v_i, \quad m_i \frac{d}{dt} v_i = -\frac{1}{N}\sum_{j\neq i} \nabla\Phi_R(x_i - x_j).$$

The potential is scaled by R as it refers to the range of interaction. $\Phi_R(x) = \Phi(x/R)$, where the potential Φ is usually radially symmetric. The hard sphere interaction case can be viewed as a special case where

$$\Phi(x) = \begin{cases} \infty & |x| \leq 1 \\ 0 & |x| > 1 \end{cases}.$$

For the case of long range interaction, namely $R \sim 1$. One can take the BBGKY hierarchy, apply chaotic assumption and reach a *Vlasov-type equation*

$$\partial_t f + v \cdot \nabla_x f + F \cdot \nabla_v f = 0, \tag{1.5}$$

where the interaction force F is given by

$$F(t,x) = -\int_{\Omega \times \mathbb{R}^D} \nabla \Phi(x-y) f(t,y,v_*) dy dv_*.$$

Exercise 4. *Formally derive equation* (1.5).

Note that the rigorous validation of the chaotic assumption is open when Φ is singular, e.g. Newtonian potential, it corresponds to Vlasov-Poisson equation in plasma physics.

The Boltzmann setup involves short range interactions with $R \ll 1$, with Φ_R is supported in a ball of radius R, or $\Phi_R(x)$ is very small when $|x| > R$. For the latter case, one can apply a Grad cutoff to restrict the interactions to the ball.

We aim to use the same argument in the hard sphere case to derive Boltzmann equation with soft potential.

Let us first observe some concepts where soft potential differs from hard sphere case:

- The interaction is not instant. However, the time for each binary interaction is of order $\mathcal{O}(R)$, which vanishes as $R \to 0$.
- Multiple interactions are not a set of zero measure in the set of all interactions. However, if $NR^D \to 0$ (which is true for Boltzmann-Grad scaling), the measure tends to zero. Therefore, we can still consider binary interactions only.
- The locations for pre-collision and post-collision for a particle are not the same. The boundary condition should be $f(t,\mathbf{x}',\mathbf{v}') = f(t,\mathbf{x},\mathbf{v})$. Of course the difference between $\mathbf{x}'$ and $\mathbf{x}$ is of order $\mathcal{O}(R)$, which vanishes as $R \to 0$. However, n_{ij} changes in the duration of the interaction by order $\mathcal{O}(1)$. One needs to understand the relationship between n_{ij} and ω.

Now, we focus on the microscopic binary interaction when two particles are within distance R. The dynamics read

$$m_1 \ddot{x}_1 = -\nabla_{x_1} \Phi_R(x_1 - x_2), \quad m_2 \ddot{x}_2 = -\nabla_{x_2} \Phi_R(x_2 - x_1).$$

Using center of mass coordinates $X = (m_1 x_1 + m_2 x_2)/M$ and $x = x_2 - x_1$, we get

$$M\ddot{X} = 0, \quad m\ddot{x} = -\nabla_x \Phi_R(x).$$

The first equation characterizes conservation of momentum. The second equation is a Hamiltonian system in relative location and velocity $(x(t), v(t))$.

Exercise 5. *Prove the following conservations for the (x, v) dynamics:*

1. *Conservation of energy:* $\frac{d}{dt}E = \frac{d}{dt}\left(\frac{1}{2}m|v|^2 + \Phi_R(x)\right) = 0,$
2. *Conservation of angular momentum:* $\frac{d}{dt}\mathbb{L} := \frac{d}{dt}(v \wedge x) = \frac{d}{dt}\left(vx^T - xv^T\right) = 0,$ *where $\mathbb{L}$ is a $D \times D$ anti-symmetric matrix.*

One can also check another identity

$$\mathbb{L}^3 + l^2\mathbb{L} = 0, \quad \textit{where } l^2 = |x|^2|v|^2 - (x \cdot v)^2 \geq 0.$$

The relative trajectory of the collision is illustrated in Figure 1.2.

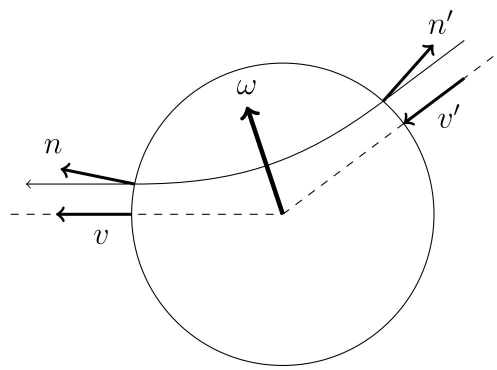

Figure 1.2: Illustration of the relative trajectory of a collision

Let us write $x(t) = r(t)n(t)$, where $r(t) = |x(t)|$ is the relative distance and $n(t) = x(t)/r(t)$ is the direction. We shift the time so that $n(0) = \omega$ for convenience. By symmetry, we know two particles enter the interaction zone at time $t = -t_*$ and left at $t = t_*$. Calculate

$$v(t) = \dot{x}(t) = \dot{r}(t)n(t) + r(t)\dot{n}(t),$$

where

$$\dot{n}(t) = \frac{v(t)}{r(t)} - \frac{(x(t) \cdot v(t))x(t)}{r(t)^3} = \frac{1}{r(t)^2}\mathbb{L}n(t).$$

Therefore, we can calculate $n = n(t_*)$ by

$$n = n(t_*) = e^{\int_0^{t_*} \frac{1}{r(t)^2} dt \mathbb{L}} \omega.$$

if $r(t)$ and t_* is known.

Next, we solve $r(t)$ by conservation of energy. Calculate

$$|v(t)|^2 = \dot{r}(t) n(t) \cdot v(t) + r(t) \dot{n}(t) \cdot v(t) = \dot{r}(t)^2 + \frac{l^2}{r(t)^2}.$$

Note that Φ_R is radial, namely $\Phi_R(x) = \Phi_R(r)$. Also, $\Phi_R(r) = 0$ when $r > R$. Then

$$\frac{1}{2} m |v(t)|^2 + \phi_R(x) = \frac{1}{2} m |v|^2.$$

It yields a first order ODE of $r(t)$:

$$\dot{r}(t)^2 = -\frac{l^2}{r(t)^2} + |v|^2 - \frac{2}{m} \Phi_R(r(t)),$$

and one can get an implicit expression of $r(t)$.

$$t = \int_{r_0}^{r(t)} \frac{1}{\sqrt{-\frac{l^2}{r^2} + |v|^2 - \frac{2}{m} \Phi_R(r)}} dr.$$

By symmetry, we know $\dot{r}(0) = 0$, therefore $r_0 = r(0)$ solves

$$-\frac{l^2}{r_0^2} + |v|^2 - \frac{2}{m} \Phi_R(r_0) = 0.$$

Finally, t_* can be determined by $r(t_*) = R$.

Although in general one can not express the collision operator explicitly, it can be written in the form of (1.4). Moreover, the collisional cross-section only depends on relative speed $|v|$ and the deflection parameter ω. Its value is non-negative.

1.3 Properties of the Boltzmann equation

Let us recall the Boltzmann equation we derived,

$$\partial_t f + v \cdot \nabla_x f = \mathcal{B}[f, f] \tag{1.6}$$

with binary collision operator

$$\mathcal{B}[f, f](t, x, v) = \int_{\mathbb{S}^{D-1} \times \mathbb{R}^D} b(|v_* - v|, \omega) \big(f(t, x, v') f(t, x, v'_*) - f(t, x, v) f(t, x, v_*) \big) d\omega dv_*,$$

where

$$v' = v - \omega \cdot (v - v_*) \omega, \quad v'_* = v_* + \omega \cdot (v - v_*) \omega.$$

The collision operator is spatial homogeneous and invariant under Galilean transformation in velocity variable.

Exercise 6. *Let the shifting operator $\mathcal{T}_u f(v) = f(v-u)$ and rotation operator $\mathcal{T}_O f(v) = f(O^T v)$, where O is an orthonormal matrix. Prove that*

$$\mathcal{B}[\mathcal{T}_u f, \mathcal{T}_u f] = \mathcal{T}_u \mathcal{B}[f,f], \quad \mathcal{B}[\mathcal{T}_O f, \mathcal{T}_O f] = \mathcal{T}_O \mathcal{B}[f,f].$$

1.3.1 Local conservation laws

We first state the *Boltzmann identity.* Fix an $x \in \Omega$, Giving any function $\xi = \xi(v)$, using symmetry and elasticity of the collision, we get

$$\begin{aligned}\int_{\mathbb{R}^D} \xi(v)\mathcal{B}[f,f](t,x,v)dv &= \frac{1}{2}\int_{\mathbb{S}^{D-1}\times\mathbb{R}^{2D}} \big(\xi(v)+\xi(v_*)\big)b(|v_*-v|,\omega)(f'f'_* - ff_*)d\omega dv_* dv \\ &= \frac{1}{4}\int_{\mathbb{S}^{D-1}\times\mathbb{R}^{2D}} \big(\xi(v)+\xi(v_*)-\xi(v')-\xi(v'_*)\big)b(|v_*-v|,\omega)(f'f'_* - ff_*)d\omega dv_* dv.\end{aligned}$$

We call ξ is *collision invariant* if

$$\xi(v)+\xi(v_*)-\xi(v')-\xi(v'_*) = 0.$$

Simple examples include $\xi(v) = 1, v_1, \cdots, v_D, |v|^2$. In fact, we have the following lemma.

Lemma 1.1. *Let $\xi : \mathbb{R}^D \to \mathbb{R}$ be a $\mathcal{C}^2$ function. If ξ is a collision invariant, then*

$$\xi(v) = a + b\cdot v + c|v|^2$$

for some constants $a, c \in \mathbb{R}, b \in \mathbb{R}^D$.

Proof. The constants $a = \xi(0), b = \nabla\xi(0)$ and $c = \frac{1}{2D}\Delta\xi(0)$. We need to show that

$$\eta(v) = \xi(v) - a - b\cdot v - c|v|^2 \equiv 0,$$

where η is a collision invariant with $\eta(0) = 0, \nabla\eta(0) = 0$ and $\Delta\eta(0) = 0$.

Take (v, v_*, ω) such that $v\cdot v_* = 0$ and $\omega = \frac{v}{|v|}$. Then, $v' = 0$ and $v'_* = v + v_*$. By collision invariant, we have

$$\eta(v)+\eta(v_*) = \eta(0)+\eta(v+v_*) \quad \Rightarrow \quad \eta(r\omega)-\eta(0) = \eta(v_*+r\omega)-\eta(v_*),$$

where $r = |v|$. Take the limit $r \to 0^+$, we obtain directional limits

$$\omega\cdot\nabla\eta(0) = \omega\cdot\nabla\eta(v_*).$$

Since $\nabla\eta(0) = 0$, we obtain $v\cdot\nabla\eta(v_*) = 0$, for all v that is orthogonal to v_*. This implies that η is a radial function.

Next, we pick $v = rn$ and $v_* = sn$ where $r, s \in \mathbb{R}$ and $n \in \mathbb{S}^{D-1}$. So v is parallel to v_*. Pick ω appropriately so that $(v' - v'_*)\cdot n = 0$. It is easy to check that

$$|v'| = |v'_*| = \sqrt{\frac{r^2+s^2}{2}}.$$

By collision invariant and η is radial, we get

$$\eta(r) + \eta(s) = 2\eta\left(\sqrt{\frac{r^2+s^2}{2}}\right).$$

Fix $r > 0$, take second derivative in s and evaluate at $s = 0$:

$$0 = \frac{d^2}{ds^2}\eta(s)\Big|_{s=0} = \frac{\sqrt{2}}{r}\frac{d}{ds}\eta\left(\sqrt{\frac{r}{2}}\right).$$

This implies $\frac{d}{dr}\eta = 0$, and therefore $\eta \equiv 0$. □

For simplicity, let us denote

$$\langle g \rangle = \int_{\mathbb{R}^D} g(v)dv.$$

For spatial homogeneous Boltzmann equation

$$\partial_t f = \mathcal{B}[f, f], \tag{1.7}$$

if ξ is collision invariant, one has

$$\langle \xi f \rangle = \int_{\mathbb{R}^D} \xi(v) f(t, v)dv = \int_{\mathbb{R}^D} \xi(v)\mathcal{B}[f, f](t, v)dv = 0.$$

This leads to locally conserved quantities. In particular, we have

- Conservation of mass: $\xi(v) = 1$, $\rho(t) := \langle f \rangle = \rho(0)$.
- Conservation of momentum: $\xi(v) = v$, $j(t) := \langle vf \rangle = j(0)$.
- Conservation of energy: $\xi = \frac{1}{2}|v|^2$, $E(t) := \frac{1}{2}\langle |v|^2 f\rangle = E(0)$.

Some other physically relavent quantities are also conserved, including

- Average velocity: $u(t) := j(t)/\rho(t) = u(0)$.
- Kinetic energy: $E_K(t) = \frac{1}{2}\rho(t)u(t)^2 = E_K(0)$.
- Internal of energy: $e(t) = \frac{1}{2}\langle (v-u)^2 f\rangle = e(0)$. One can check $E(t) = E_K(t) + e(t)$.
- Temperature: $\theta(t) = 2e(t)/(D\rho(t)) = \theta(0)$.

For Boltzmann equation (1.6), we say $\xi(t, x, v)$ is *locally conserved* if

$$\partial_t \int_{\mathbb{R}^D} \xi(t, x, v) f(t, x, v)dv + \nabla_x \cdot \int_{\mathbb{R}^D} v\xi(t, x, v) f(t, x, v)dv = 0.$$

Let us multiply (1.6) by ξ and integrate in v:

$$\int_{\mathbb{R}^D}(\partial_t + v\cdot\nabla_x)(\xi f)dv - \underbrace{\int_{\mathbb{R}^D}(\partial_t\xi + v\cdot\nabla_x\xi)f dv}_{I} = \underbrace{\int_{\mathbb{R}^D}\xi\mathcal{B}[f,f]dv}_{II}.$$

To get $I = 0$, one needs $\partial_t\xi + v\cdot\nabla_x\xi = 0$, and so $\xi(t,x,v) = \xi_0(x - vt, v)$; and to get $II = 0$, one needs $\xi(t,x,\cdot)$ to be collision invariant. Therefore, all locally conserved quantities are given as follows.

$$\xi = \text{span}\{1, v, |v|^2, x - vt, v\wedge x, v\cdot(x - vt), |x - vt|^2\}.$$

For $D = 3$, the dimension of the locally conserved quantities is 13.

If we neglect boundary conditions in x (e.g. take $\Omega = \mathbb{T}^D$), then we obtain

$$\frac{d}{dt}\int_{\Omega\times\mathbb{R}^D}\xi(t,x,v)f(t,x,v)dxdv = 0.$$

It corresponds to globally conserved quantities.

1.3.2 Boltzmann H-theorem

Another important quantities for the Boltzmann collision operator is the local dissipation

$$D(f) := -\big\langle \log f\mathcal{B}[f,f]\big\rangle = -\frac{1}{4}\int_{\mathbb{S}^{D-1}\times\mathbb{R}^{2D}}\log\left(\frac{ff_*}{f'f'_*}\right)(f'f'_* - ff_*)b\, d\omega dv_* dv \geq 0.$$

For spatial homogeneous Boltzmann equation (1.7), we obtain

$$\partial_t\int_{\mathbb{R}^D} f(t,v)\log f(t,v)dv = -D(f) \leq 0.$$

It indicates that $\langle f\log f\rangle$, known as the *local entropy*, decays in time.

Note that $D(f) = 0$ if and only if

$$f'f'_* = ff_*, \tag{1.8}$$

or equivalently, $\log f$ is collision invariant. Therefore, by Lemma 1.1 the solution for the steady state $\mathcal{B}[f,f] = 0$ has the form

$$\log f(v) = a + b\cdot v + c|v|^2 \quad\Rightarrow\quad f(v) = \exp\left[c\left(v + \frac{b}{2c}\right)^2 + \left(a - \frac{|b|^2}{4c}\right)\right].$$

So, the Maxwellian is a Gaussian distribution in v. Moreover, the coefficients (a,b,c) are related to locally conserved quantities (ρ,u,θ), and

$$f(v) = \mathcal{M}(\rho,u,\theta)(v) = \frac{\rho}{(2\pi\theta)^{D/2}}e^{-\frac{|v-u|^2}{2\theta}}. \tag{1.9}$$

Exercise 7. *Prove* (1.9)*, namely check*

$$\langle \mathcal{M}(\rho,u,\theta)\rangle = \rho, \quad \langle v\mathcal{M}(\rho,u,\theta)\rangle = \rho u, \quad \langle \frac{1}{2}|v|^2\mathcal{M}(\rho,u,\theta)\rangle = \frac{D}{2}\rho\theta.$$

For the Boltzmann equation (1.6), we define *entropy*

$$\mathcal{H}(t) = \int_{\Omega\times\mathbb{R}^D} f(t,x,v)\log f(t,x,v)dxdv.$$

Neglecting the boundary effects, we can get

$$\frac{d}{dt}\mathcal{H}(t) = \underbrace{-\int_{\Omega\times\mathbb{R}^D} \nabla_x\cdot(vf(\log f+1))dxdv}_{=0,\text{ neglecting boundary}} + \underbrace{\int_{\Omega\times\mathbb{R}^D}(\log f+1)\mathcal{B}[f,f]dxdv}_{\leq 0} \leq 0.$$

So the entropy decays in time, and this corresponds to the *Boltzmann H-theorem.*

1.3.3 Relative entropy

To obtain a bound on the entropy production, we introduce the *relative entropy*

$$\mathcal{H}[f|g](t) := \int_{\Omega\times\mathbb{R}^D}\left[f\log\left(\frac{f}{g}\right) - f + g\right]dxdv, \quad \text{for } f\geq 0,\ g>0.$$

Note that $\mathcal{H}[f|g]$ can be written as

$$\mathcal{H}[f|g] = \int_{\Omega\times\mathbb{R}^D}\psi\left(\frac{f}{g}\right)gdxdv, \quad \psi(z) = z\log(z) - z + 1.$$

The function ψ is nonnegative and convex for all $z\geq 0$, and $\psi(x)=0$ if and only if $x=1$. Therefore, $\mathcal{H}[f|g]\geq 0$, and zero is attained only when $f=g$.

For practical use, we usually take $g=\mathcal{M}(\rho,u,\theta)$. In this case, we have

$$\mathcal{H}[f|\mathcal{M}] = \mathcal{H}[f] - \underbrace{\int_{\Omega\times\mathbb{R}^D} f\log\mathcal{M}dxdv}_{I} - \underbrace{\int_{\Omega\times\mathbb{R}^D} fdxdv}_{II} + \underbrace{\int_{\Omega\times\mathbb{R}^D}\mathcal{M}dxdv}_{III}.$$

Since $\log\mathcal{M}\in\text{span}\{1,v,|v|^2\}$, we have I is conserved in time. Clearly, II and III are conserved in time as well. Hence,

$$\mathcal{H}[f|\mathcal{M}](t) - \mathcal{H}[f|\mathcal{M}](0) = \mathcal{H}[f](t) - \mathcal{H}[f](0) = -\int_\Omega D(f)dx \leq 0.$$

As the relative entropy is nonnegative, we obtain the entropy production estimate

$$\frac{1}{4}\int_0^\infty dt\int_\Omega dx\int_{\mathbb{S}^{D-1}\times\mathbb{R}^{2D}}\log\left(\frac{f'f'_*}{ff_*}\right)(f'f'_* - ff_*)b\,d\omega dv_*dv \leq \mathcal{H}[f|\mathcal{M}](0). \quad (1.10)$$

Chapter 2

Formal derivations to hydrodynamic limits

2.1 Nondimensional analysis

Consider the Cauchy problem of Boltzmann equation (1.6) with initial condition

$$f(0, x, v) = f^{in}(x, v).$$

Take $\Omega = \mathbb{T}^D$ to avoid boundary conditions as of now.

There are several quantities carry the scales of the system:

- Size: $l_A^D = \int_{\mathbb{T}^D} dx$,
- Mass: $\rho_A \lambda_A^D = \int_{\mathbb{T}^D \times \mathbb{R}^D} f^{in} dx dv$,
- Thermal speed $\sim \theta_A^{1/2}$: $\frac{D}{2} \rho_A l_A^D \theta_A = \int_{\mathbb{T}^D \times \mathbb{R}^D} \frac{1}{2} |v|^2 f^{in} dx dv$,
- collision rate: $\frac{\rho_A}{\mu_A} = \int_{\mathcal{S}^{D-1} \times \mathbb{R}^{2D}} \mathcal{M}(\rho_A, 0, \theta_A)(v) \mathcal{M}(\rho_A, 0, \theta_A)(v_*) b \; d\omega dv_* dv$.

μ_A represents the scale of the average time that particles in the equilibrium density $\mathcal{M}(\rho_A, 0, \theta_A)$ spend traveling freely between two collisions. So, the mean free path

$$\lambda \sim \theta_A^{1/2} \mu_A.$$

Now, we can perform a non-dimensionization of the system

$$v = \theta_A^{1/2} \hat{v}, \quad x = l_A \hat{x}, \quad t = \tau_A \hat{t}, \quad f(t, x, v) = \frac{\rho_A}{\theta_A^{D/2}} \hat{F}(\hat{t}, \hat{x}, \hat{v}).$$

Then, we get

$$\partial_t f = \frac{\rho_A}{\tau_A \theta_A^{D/2}} \partial_{\hat{t}} \hat{f}, \quad v \cdot \nabla_x f = \frac{\rho_A}{l_A \theta_A^{\frac{D-1}{2}}} \hat{v} \cdot \nabla_{\hat{x}} \hat{f}, \quad \mathcal{B}[f,f] = \frac{\rho_A}{\mu_A \theta_A^{D/2}} \hat{\mathcal{B}}(\hat{f}, \hat{f}).$$

We reach the following non-dimensionized equation,

$$\mathrm{St} \partial_{\hat{t}} \hat{f} + \hat{v} \cdot \nabla_{\hat{x}} \hat{f} = \frac{1}{\mathrm{Kn}} \hat{\mathcal{B}}(\hat{f}, \hat{f}),$$

where the *Strouhal number* and *Knudsen number* are given as

$$\mathrm{St} = \frac{l_A}{\theta_A^{1/2} \tau_A}, \quad \mathrm{Kn} = \frac{\theta_A^{1/2} \mu_A}{l_A}.$$

Two other important quantities are called *Mach number* and *Reynorlds number*, given as defined as

$$\mathrm{Ma} = \frac{u_A}{\theta_A^{1/2}}, \quad \mathrm{Re} = \frac{\mathrm{Ma}}{\mathrm{Kn}}.$$

Here u_A is the scale for the macroscopic velocity. A natural choices of scales is to take $l_A = \tau_A u_A$. In this case, Mach number is the same as Strouhal number. However, for some perturbative regimes, u_A is very small compared with l_A/τ_A. So Mach number could be much smaller than Strouhal number. As we will see later, this regime will lead to incompressible fluid limits.

Let us work with the non-dimensionized equation from now on, dropping the hats. Different choices of St and Kn corresponds to different hydrodynamic limits.

2.2 Compressible Euler limit

2.2.1 Formal derivation of global Maxwellian

We set St$= 1$ and Kn$= \epsilon \ll 1$. The equation reads

$$\partial_t f_\epsilon + v \cdot \nabla_x f_\epsilon = \frac{1}{\epsilon} \mathcal{B}[f_\epsilon, f_\epsilon].$$

If we can formally assume $f_\epsilon \to f$. Then, $\mathcal{B}[f,f] = 0$ and the limiting solution has the form $f(t,x,v) = \mathcal{M}(\rho(t,x), u(t,x), \theta(t,x))(v)$.

To determine (ρ, u, θ), we apply local conservation laws and get

$$\begin{cases} \partial_t \langle f \rangle + \nabla_x \cdot \langle v f \rangle = 0 \\ \partial_t \langle v f \rangle + \nabla_x \cdot \langle v \otimes v f \rangle = 0 \\ \partial_t \langle \frac{1}{2} |v|^2 f \rangle + \nabla_x \cdot \langle \frac{1}{2} v |v|^2 f \rangle = 0 \end{cases} \tag{2.1}$$

The first equation is the *continuity equation*

$$\partial_t \rho + \nabla_x \cdot (\rho u) = 0.$$

The second equation is the *momentum equation*

$$\partial_t (\rho u) + \nabla_x \cdot (\rho u \otimes u + \mathbb{P}) = 0,$$

where $\mathbb{P}$ is the *pressure tensor* defined as

$$\mathbb{P} = \langle (v-u) \otimes (v-u) f \rangle.$$

The third equation can be written in terms of temperature θ

$$\partial_t \left(\frac{1}{2}\rho|u|^2 + \frac{D}{2}\rho\theta \right) + \nabla_x \cdot \left(\frac{1}{2}\rho|u|^2 u + \frac{D}{2}\rho\theta u + \mathbb{P}u + \frac{1}{2}\langle (v-u)|v-u|^2 f \rangle \right) = 0.$$

Note that the macroscopic system is not closed. Since we know that the limiting solution is a Maxwellian distribution, we can plug in the ansatz and close the system. Indeed, we have

$$\begin{aligned}
&\nabla_x \cdot \mathbb{P} = \nabla_x \cdot \langle v \otimes v \mathcal{M}(\rho, 0, \theta) \rangle = \nabla_x (\rho\theta) = \nabla_x \cdot (\rho\theta\mathbb{I}), \\
&\nabla_x \cdot (\mathbb{P}u) = \nabla_x \cdot (\langle v \otimes v \mathcal{M}(\rho, 0, \theta) \rangle u) = \nabla_x \cdot (\rho\theta u), \\
&\langle (v-u)|v-u|^2 \mathcal{M}(\rho, u, \theta) \rangle = 0.
\end{aligned}$$

Therefore, we obtain the *compressible Euler equations*

$$\begin{cases}
\partial_t \rho + \nabla_x \cdot (\rho u) = 0, \\
\partial_t (\rho u) + \nabla_x \cdot (\rho u \otimes u + \rho\theta\mathbb{I}) = 0, \\
\partial_t \left(\frac{1}{2}\rho|u|^2 + \frac{D}{2}\rho\theta \right) + \nabla_x \cdot \left(\frac{1}{2}\rho|u|^2 u + \frac{D+2}{2}\rho\theta u \right) = 0.
\end{cases}$$

Moreover, the Boltzmann H-theorem takes the form

$$\partial_t \langle f \log f \rangle + \nabla_x \cdot \langle v f (\log f + 1) \rangle \leq 0. \tag{2.2}$$

Plug in the Maxwellian distribution, we obtain the entropy decay condition for the limiting system

$$\partial_t \left(\rho \log \frac{\rho}{\theta^{D/2}} \right) + \nabla_x \cdot \left(\rho u \log \frac{\rho}{\theta^{D/2}} \right) \leq 0. \tag{2.3}$$

This is so called *Lax admissibility condition*, which is useful in selecting solutions of compressible Euler equations that satisfy second principle of thermodynamics.

We call distribution $\mathcal{M}(\rho, u, \theta)$ where (ρ, u, θ) solves the compressible Euler equations as *global Maxwellian*.

2.2.2 The moment method

The moment method provides a slightly more rigorous derivation of compressible Euler limit, assuming the solution of Boltzmann equation is well-behaved.

Consider the following initial value problem

$$\partial_t f_\epsilon + v \cdot \nabla_x f_\epsilon = \frac{1}{\epsilon}\mathcal{B}[f_\epsilon, f_\epsilon], \quad f_\epsilon^{in} = \mathcal{M}(\rho^{in}, u^{in}, \theta^{in}).$$

Suppose the initial condition $\rho^{in} \geq 0$ and $\theta^{in} > 0$ almost everywhere. Assume the solution f_ϵ exists and $f_\epsilon \to f$ almost everywhere. We will discuss what recipes are needed to make the argument rigorous.

Step 1: f is a local Maxwellian.

Consider the ralative entropy $\mathcal{H}[f_\epsilon|\mathcal{M}]$, where we choose $\mathcal{M}$ to be independent of ϵ and t. For instance, we can take $\mathcal{M} = \mathcal{M}(\bar{\rho}, \bar{u}, \bar{\theta})$, where

$$\begin{bmatrix} \bar{\rho} \\ \bar{\rho}\bar{u} \\ \frac{1}{2}\bar{\rho}|\bar{u}|^2 + \frac{D}{2}\bar{\rho}\bar{\theta} \end{bmatrix} = \frac{1}{|\Omega|}\int_\Omega \begin{bmatrix} \rho^{in} \\ \rho^{in}u^{in} \\ \frac{1}{2}\rho^{in}|u^{in}|^2 + \frac{D}{2}\rho^{in}\theta^{in} \end{bmatrix} dx.$$

The entropy production bound (1.10) states

$$\int_\Omega D(f^\epsilon)dx \leq \epsilon\mathcal{H}[f^{in}|\mathcal{M}].$$

Apply Fatou's lemma, we get

$$0 \leq \int_0^T \int_\Omega D(f)dxdt \leq \liminf_{\epsilon\to 0} \int_0^T \int_\Omega D(f^\epsilon)dxdt \leq \liminf_{\epsilon\to 0} \epsilon\mathcal{H}[f^{in}|\mathcal{M}] = 0.$$

Therefore, $D(f) = 0$ almost everywhere, and f has to be a local Maxwellian $f = \mathcal{M}(\rho, u, \theta)$.

To ensure the argument to work, we need to assume the initial data satisfy that $\mathcal{H}[f^{in}|\mathcal{M}]$ is finite.

Step 2: (ρ, u, θ) is a solution of the compressible Euler system.

The compressible Euler equation is derived by taking $f = \mathcal{M}(\rho, u, \theta)$ to the local conservation laws (2.1). Therefore, we are able to determine the dynamics of (ρ, u, θ) as long as we can pass the limit in the local conservation laws, namely

$$\begin{aligned}
&\langle f_\epsilon\rangle \to \langle f\rangle = \rho, \\
&\langle vf_\epsilon\rangle \to \langle vf\rangle = \rho u, \\
&\langle v\otimes vf_\epsilon\rangle \to \langle v\otimes vf\rangle = \rho u\otimes u + \rho\theta\mathbb{I}, \\
&\langle v|v|^2 f_\epsilon\rangle \to \langle v|v|^2 f\rangle = \rho|u|^2 u + (D+2)\rho\theta u.
\end{aligned}$$

To prove these convergences are not trivial and we won't discuss here.

Step 3: Recover Lax admissibility condition (2.3)**.**

The Lax admissibility condition is obtained by taking $f = \mathcal{M}(\rho, u, \theta)$ to the Boltzmann H-theorem (2.2). So, similar as the step 2, the condition (2.3) is satisfied if we can pass the limit of the following quantities.

$$\langle f_\epsilon \log f_\epsilon \rangle \to \langle f \log f \rangle = \rho \log\left(\frac{\rho}{\theta^{D/2}}\right) - \frac{D}{2}(1 + \log(2\pi))\rho,$$
$$\langle v f_\epsilon \log f_\epsilon \rangle \to \langle v f \log f \rangle = \rho u \log\left(\frac{\rho}{\theta^{D/2}}\right) - \frac{D}{2}(1 + \log(2\pi))\rho u.$$

2.3 Compressible Navier-Stokes corrections

2.3.1 Hilbert expansion

One other way to derive compressible Euler limit is to formally expand the solution as a series of approximated solutions

$$f = f_0 + \epsilon f_1 + \epsilon^2 f_2 + \cdots = f_0(1 + \epsilon g_1 + \epsilon^2 g_2 + \cdots),$$

where ϵ is the Knudsen number which is small, and tend to 0 in the limit. The *Hilbert expansion* is one of the expansion proposed by Hilbert in 1912.

We perform an asymptotic analysis on Boltzmann equation

$$\partial_t f + v \cdot \nabla_x f = \frac{1}{\epsilon}\mathcal{B}[f, f].$$

The $\mathcal{O}(\epsilon^{-1})$ term reads $\mathcal{B}[f_0, f_0] = 0$. So, $f_0 = \mathcal{M}(\rho_0, u_0, \theta_0) =: \mathcal{M}_0$.

For $n \geq 0$, the $\mathcal{O}(\epsilon^n)$ term has the form

$$\partial_t f_n + v \cdot \nabla_x f_n = 2\mathcal{B}[\mathcal{M}_0, f_{n+1}] + \sum_{i=1}^{n} \mathcal{B}[f_i, f_{n+1-i}].$$

Let us first suppose $\{f_i\}_{i=1}^n$ is given. To solve for f_{n+1}, we need to invert the *linearized collision operator*

$$\mathcal{L}_{\mathcal{M}_0}[g](t, x, v) = \frac{2}{\mathcal{M}_0}\mathcal{B}[\mathcal{M}_0, \mathcal{M}_0 g] = \int_{\mathbb{S}^{D-1}\times\mathbb{R}^D} \mathcal{M}_{0*}(g'_* + g' - g_* - g)b \, d\omega dv_*. \quad (2.4)$$

The second equality can be obtained using (1.8): $\mathcal{M}_0\mathcal{M}_{0*} = \mathcal{M}'_0\mathcal{M}'_{0*}$.

Then, f_{n+1} can be expressed as

$$f_{n+1} = \mathcal{M}_0 \mathcal{L}_{\mathcal{M}_0}^{-1}\left[\frac{1}{\mathcal{M}_0}\left(\partial_t f_n + v \cdot \nabla_x f_n - \sum_{i=1}^{n} \mathcal{B}[f_i, f_{n+1-i}]\right)\right].$$

Exercise 8. *Prove that $\mathcal{L}_{\mathcal{M}_0}$ is self-adjoint under $L^2_{\mathcal{M}_0}$ inner product $(\cdot, \cdot)_{\mathcal{M}_0}$:*

$$(f, g)_{\mathcal{M}_0} = \int_{\mathbb{R}^D} f(v)\bar{g}(v)\mathcal{M}_0(v)dv.$$

Lemma 2.1. $\mathcal{L}_{\mathcal{M}_0}[g](v) \equiv 0$ *if and only if g is collision invariant.*

Proof. If g is collision invariant, the definition (2.4) directly implies $\mathcal{L}_{\mathcal{M}_0}[g] = 0$.

To get the other direction, we first apply Boltzmann identity on $\mathcal{B}[\mathcal{M}_0, f]$ and get

$$\langle \xi(v)\mathcal{B}[\mathcal{M}_0, f]\rangle = \frac{1}{8}\int_{\mathbb{S}^{D-1}\times\mathbb{R}^{2D}} \left(\xi+\xi_*-\xi'-\xi'_*\right)(f'\mathcal{M}'_{0*}+f'_*\mathcal{M}'_0-f\mathcal{M}_{0*}-f_*\mathcal{M}_0)b\, d\omega dv_* dv.$$

Take $\xi(v) = \bar{g}(v)$ and $f = \mathcal{M}_0 g$. Denote $(\cdot,\cdot)$ as L^2 inner product. We obtain

$$(g, \mathcal{L}_{\mathcal{M}_0}[g])_{\mathcal{M}_0} = \langle \bar{g}\mathcal{B}[\mathcal{M}_0, \mathcal{M}_0 g]\rangle = -\frac{1}{4}\int_{\mathbb{S}^{D-1}\times\mathbb{R}^{2D}} \left|g+g_*-g'-g'_*\right|^2 \mathcal{M}_0\mathcal{M}_{0*} b\, d\omega dv_* dv \le 0.$$

In particular, $(g, \mathcal{L}_{\mathcal{M}_0}[g])_{\mathcal{M}_0} = 0$ if and only if $g + g_* = g' + g'_*$, namely g is collision invariant.

If $\mathcal{L}_{\mathcal{M}_0}[g](v) \equiv 0$, it is clear that $(g, \mathcal{L}_{\mathcal{M}_0}[g])_{\mathcal{M}_0} = 0$ and therefore g is collision invariant. □

Given any g, we know that

$$(\xi, \mathcal{L}_{\mathcal{M}_0}[g])_{\mathcal{M}_0} = (\mathcal{L}_{\mathcal{M}_0}[\xi], g)_{\mathcal{M}_0} = 0$$

if and only if $\xi \in \text{span}\{1, v, |v|^2\} =: \mathcal{N}$. Therefore, the range of $\mathcal{L}_{\mathcal{M}_0}$ is the space which is orthogonal to $\mathcal{N}$. Denote $\mathcal{R} = \mathcal{N}^{\perp_{\mathcal{M}_0}}$, where the orthogonality is with respect to $(\cdot,\cdot)_{\mathcal{M}_0}$.

We claim that one can apply Fredholm alternative theory and obtain that $\mathcal{L}_{\mathcal{M}_0}[g] = h$ has a unique solution in $\mathcal{R}$ if $h \in \mathcal{R}$. We will pause the related discussion to Section 2.3.2.

Let us start with $n = 0$. Denote $\{\xi_\alpha\}_{i=1}^{D+2}$ be $(1, v_1, \cdots, v_D, \frac{1}{2}|v|^2)$. $\mathcal{L}_{\mathcal{M}_0}$ is invertible if

$$\frac{1}{\mathcal{M}_0}(\partial_t\mathcal{M}_0 + v\cdot\nabla_x\mathcal{M}_0) \in \mathcal{R}, \quad \text{or equivalently} \quad \partial_t\langle\xi_\alpha\mathcal{M}_0\rangle + \nabla_x\cdot\langle\xi_\alpha v\mathcal{M}_0\rangle = 0.$$

By the definition of $\mathcal{N}$, we conclude that $\mathcal{M}_0$ has to satisfy compressible Euler equation. More detailed calculation will be given in Section 2.3.3.

Next, let us solve for the first order correction $f_1 = \mathcal{M}_0 g_1$. We shall decompose itg_1 into two parts $g_{11} \in \mathcal{R}$ and $g_{12} \in \mathcal{N}$. g_{11} can be obtained by inverting the linearized operator

$$g_{11} = \mathcal{L}_{\mathcal{M}_0}^{-1}\left[\frac{1}{\mathcal{M}_0}(\partial_t\mathcal{M}_0 + v\cdot\nabla_x\mathcal{M}_0)\right].$$

g_{12} can be determined by the solvability condition on g_2. Take $n = 1$. g_2 is solvable if

$$\frac{1}{\mathcal{M}_0}(\partial_t f_1 + v\cdot\nabla_x f_1 - \mathcal{B}[f_1, f_1]) \in \mathcal{R}, \quad \text{or equivalently} \quad \partial_t\langle\xi_\alpha f_1\rangle + \nabla_x\cdot\langle\xi_\alpha v f_1\rangle = 0. \tag{2.5}$$

Indeed, we can express $g_{12}(v)$ as

$$g_{12} = \sum_\alpha c_\alpha \xi_\alpha(v).$$

To determine c_α, we use solvability condition (2.5) and get

$$\partial_t \rho_1^\alpha + \nabla_x \cdot j_1^\alpha = 0, \tag{2.6}$$

where $\rho^\alpha = \langle \xi_\alpha f \rangle$ represents conserved quantities $(\rho, \rho u, \mathcal{E})$, and j^α is the corresponding flux.

We claim that $\{\rho_1^\alpha\}$ determines $\{c_\alpha\}$. Indeed,

$$\rho_1^\alpha = \langle \xi_\alpha f_1 \rangle = \langle \xi_\alpha g_{11} M_0 \rangle + \sum_{\beta=1}^{D+2} c_\beta \langle \xi_\alpha \xi_\beta \mathcal{M}_0 \rangle.$$

Exercise 9. *Show that the matrix $\langle \xi_\alpha \xi_\beta \mathcal{M}_0 \rangle$ is nonsingular, and so c_α can be expressed in terms of ρ_1^α.*

Therefore, as long as we can solve ρ_1^α from (2.6), g_{12} can be determined. One can continue the procedure to get higher order corrections of the solution.

To solve ρ_1^α however, is not trivial, as (2.6) is not closed. We can write $E^\alpha(\rho^\beta) = S^\alpha$, where $E^\alpha(\rho^\beta) = 0$ represents compressible Euler equations, and

$$S^\alpha = \begin{bmatrix} 0 \\ -\nabla_x \cdot (\mathbb{P} - \rho\theta\mathbb{I}) \\ -\frac{1}{2}\nabla_x \cdot (\mathbb{P}u - \rho\theta u + \langle (v-u)|v-u|^2 f \rangle) \end{bmatrix} = \begin{bmatrix} 0 \\ -\nabla_x \cdot \langle A(v-u) f \rangle \\ -\nabla_x \cdot \langle (A(v-u)u + B(v-u)) f \rangle \end{bmatrix},$$

with

$$A(z) = z \otimes z - \frac{1}{D}|z|^2, \quad B(z) = \frac{1}{2}(|z|^2 - (D+2))z.$$

Here, Since $\rho^\alpha = \sum_{i=0}^\infty \rho_i^\alpha$, one can formally expand E^α and S^α as

$$E^\alpha = \sum_{i=0}^\infty \epsilon^i E_i^\alpha, \quad S^\alpha = \sum_{i=0}^\infty \epsilon^i S_i^\alpha.$$

Then, ρ_i^α can be obtained by solving $E_i^\alpha = S_i^\alpha$.

In particular, $E_0^\alpha = E^\alpha(\rho_0^\alpha)$, and $S_0 = 0$. Therefore, ρ_0 solves compressible Euler equations. ρ_1^α provides a correction to compressible Euler system. Higher order corrections can be made through a similar procedure.

A major question for Hilbert expansion is that $\{\rho_i^\alpha\}$ is not necessarily uniform in ϵ. Therefore, the expansion is not valid in many physically relavent circumstances.

2.3.2 Fredholm alternative for the linearized collision operator

In this section, we focus on the properties of the linearized collision operator $\mathcal{L}_{\mathcal{M}_0}$.

Let us take the Hilbert decompose of $\mathcal{L}_{\mathcal{M}_0}$ as follows:

$$\mathcal{L}_{\mathcal{M}_0}[g](v) = -\nu(v)g(v) + \mathcal{K}[g](v) = -\nu(v)g(v) - \mathcal{K}_1[g](v) + \mathcal{K}_2[g](v) + \mathcal{K}_3[g](v),$$

where the *collision frequency* ν and integral operators $\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3$ are defined as

$$\begin{aligned} \nu(v) &= \int_{\mathcal{S}^{D-1}\times\mathbb{R}^D} \mathcal{M}_{o*} b \, d\omega dv_*, \quad \mathcal{K}_1[g](v) = \int_{\mathcal{S}^{D-1}\times\mathbb{R}^D} g_* \mathcal{M}_{o*} b \, d\omega dv_*, \\ \mathcal{K}_2[g](v) &= \int_{\mathcal{S}^{D-1}\times\mathbb{R}^D} g' \mathcal{M}_{o*} b \, d\omega dv_*, \quad \mathcal{K}_3[g](v) = \int_{\mathcal{S}^{D-1}\times\mathbb{R}^D} g'_* \mathcal{M}_{o*} b \, d\omega dv_*. \end{aligned} \tag{2.7}$$

Consider the case where b is not very singular so that ν is bounded.

Exercise 10. *Prove that $\mathcal{K}_j$ is a bounded operator on $L^2(\mathcal{M}_0 dv) \to L^2(\mathcal{M}_0 dv)$, and $\|\mathcal{K}_j\|_{L^2(\mathcal{M}_0 dv)\to L^2(\mathcal{M}_0 dv)} \le 1$.*

Moreover, one can prove compactness of $\mathcal{K}_j$. For $\mathcal{K}_2$ and $\mathcal{K}_3$, the procedure is not trivial and we won't discuss it in this notes.

The compactness argument allows us to apply Fredholm alternative theory to invert $\mathcal{L}_{\mathcal{M}_0}$ from $\mathcal{R}$ to $\mathcal{R}$.

2.3.3 Chapman-Enskog expansion and Navier-Stokes correction

The Chapman-Enskog expansion is a slightly different procedure than Hilbert expansion. Since the series $\rho^\alpha = \sum_{i=0}^\infty \epsilon^i \rho_i^\alpha$ does not necessarily converge, we will keep ρ^α unexpanded. The expansion on f still reads

$$f = f_0 + \epsilon f_1 + \epsilon^2 f_2 + \cdots .$$

However, we seek for $\{f_i\}$ such that

$$\langle \xi_\alpha f_0 \rangle = \langle \xi_\alpha f^{in} \rangle = \rho^\alpha, \quad \text{and} \quad \langle \xi_\alpha f_i \rangle = 0, \ \forall\, i \ge 1.$$

So ρ^α is not expanded, and the corrections do not change the conserved quantities.

Under this setup, the conservation laws implies

$$E^\alpha(\rho^\beta) = S^\alpha = \sum_{i=0}^\infty \epsilon^i S_i^\alpha, \quad \text{where } S_i^\alpha = \begin{bmatrix} 0 \\ -\nabla_x \cdot \langle A(v-u) f_i \rangle \\ -\nabla_x \cdot \langle (A(v-u)u + B(v-u)) f_i \rangle \end{bmatrix}.$$

Like Hilbert expansion, we have $\mathcal{B}[f_0, f_0] = 0$ and implies $f_0 = \mathcal{M}_0 = \mathcal{M}(\rho, u, \theta)$. Also, $S_0^\alpha = 0$. Therefore, the conservation law can be represented as

$$E^\alpha(\rho^\beta) = 0 \qquad \text{mod } \mathcal{O}(\epsilon).$$

We reach the same compressible Euler equations. So the zeroth order of the Chapman-Enskog expansion is the same as Hilbert expansion.

The two expansions differ at order 1 in ϵ. Note that g_{11} is defined in the same way. However, for Chapman-Enskog expansion, as $\langle \xi_\alpha f_1 \rangle = 0$, then $g_{12} = 0$. Hence, we get a different first order correction of the solution $f_1 = \mathcal{M}_0 g_{11}$.

Let us try to find an explicit expression to the correction term. Calculate

$$\begin{aligned}
&\frac{1}{\mathcal{M}_0}(\partial_t\mathcal{M}_0 + v\cdot\nabla_x\mathcal{M}_0) = (\partial_t + v\cdot\nabla_x)\log\mathcal{M}_0\\
=& \frac{1}{\rho}(\partial_t + v\cdot\nabla_x)\rho - \frac{D}{2\theta}(\partial_t + v\cdot\nabla_x)\theta + \frac{v-u}{\theta}\cdot(\partial_t + v\cdot\nabla_x)u + \frac{|v-u|^2}{2\theta^2}(\partial_t + v\cdot\nabla_x)\theta\\
=& \frac{1}{\rho}(\partial_t\rho + u\cdot\nabla_x\rho + \rho\nabla_x\cdot u)\\
&+ \frac{v-u}{\theta}\cdot\left(\partial_t u + u\cdot\nabla_x u + \nabla_x\theta + \frac{\theta}{\rho}\nabla_x\rho\right)\\
&+ \left(\frac{|v-u|^2}{2\theta^2} - \frac{D}{2\theta}\right)\left(\partial_t\theta + u\cdot\nabla_x\theta + \frac{2\theta}{D}\nabla_x\cdot u\right)\\
&+ A\left(\frac{v-u}{\sqrt{\theta}}\right) : \nabla_x u + \frac{1}{\sqrt{\theta}}B\left(\frac{v-u}{\sqrt{\theta}}\right)\cdot\nabla_x\theta
\end{aligned}$$

Clearly, the first three terms lie in $\mathcal{N}$. Moreover, the following exercise shows that the last two terms lie in $\mathcal{R}$.

Exercise 11. *Prove that $A_{ij} \perp \mathcal{N}, B_k \perp \mathcal{N}$ and $A_{ij} \perp B_k$, for all $i, j, k = 1, \cdots, D$.*

One can check that the system

$$\begin{cases}
\partial_t\rho + \nabla_x\cdot(\rho u) = 0,\\
\partial_t u + u\cdot\nabla_x u + \frac{1}{\rho}\nabla_x(\rho\theta) = 0,\\
\partial_t\theta + u\cdot\nabla_x\theta + \frac{2\theta}{D}\nabla_x u = 0,
\end{cases}$$

is equivalent to compressible Euler equations. Therefore, it is clear that the solvability condition holds, and

$$g_{11} = \mathcal{L}^{-1}_{\mathcal{M}_0}\left[A\left(\frac{v-u}{\sqrt{\theta}}\right) : \nabla_x u + \frac{1}{\sqrt{\theta}}B\left(\frac{v-u}{\sqrt{\theta}}\right)\cdot\nabla_x\theta\right].$$

Lemma 2.2. *There exist scaler functions* $\mathbf{a}$ *and* $\mathbf{b}$*, such that*

$$\mathcal{L}^{-1}_{\mathcal{M}(1,0,1)}[A](v) = -\mathbf{a}(|v|^2)A(v), \quad \mathcal{L}^{-1}_{\mathcal{M}(1,0,1)}[A](v) = -\mathbf{b}(|v|^2)B(v).$$

Proof. First, we observe that A, B are invariant under rotation, namely for any orthonormal matrix O, $A(Ov) = A(v)$, $B(Ov) = B(v)$.

Let $\hat{A} = \mathcal{L}^{-1}_{\mathcal{M}(1,0,1)}[A](v) \in \mathcal{R}$. Obviously, $\hat{A}_O(v) = \hat{A}(Ov)$ is also an element of $\mathcal{R}$. Moreover, we observe

$$\mathcal{L}_{\mathcal{M}(0,1,0)}[\hat{A}_O] = A(Ov) = A.$$

By uniqueness of the solution, we know $\hat{A}_O = \hat{A}$. Hence, we are able to write $\hat{A} = -\mathbf{a}(|v|^2)A$, where $-\mathbf{a}$ is a scaler radial function. Similar argument can be applied to B. The scaler functions $\mathbf{a}$ and $\mathbf{b}$ are determined by the collision kernel. □

Let the collision cross section b has the form $b(|v-v_*|,\omega)=|v-v_*|^\gamma h(\omega)$. (Note that for hard sphere collision, $\gamma=1$.) As $\mathcal{L}_{M_0}$ is linear, we can obtain that

$$\mathcal{L}_{\mathcal{M}_0}\left[\hat{A}\left(\frac{v-u}{\sqrt{\theta}}\right)\right](v)=\rho\theta^{\gamma/2}\mathcal{L}_{\mathcal{M}(1,0,\theta)}[\hat{A}]\left(\frac{v-u}{\sqrt{\theta}}\right).$$

So, from Lemma 2.2 we know

$$\mathcal{L}^{-1}_{\mathcal{M}_0}\left[A\left(\frac{v-u}{\sqrt{\theta}}\right)\right]=-\frac{1}{\rho}\,\mathbf{a}_\theta\left(\frac{|v-u|^2}{\theta}\right)A\left(\frac{|v-u|^2}{\theta}\right),$$

where $\mathbf{a}_\theta(v)=\theta^{-\gamma/2}\mathbf{a}(v)$. For more general cross section b, the expression of $\mathbf{a}_\theta$ could be more complicated. We apply the same argument to $\mathbf{b}$ and get

$$f_1=-\frac{\mathcal{M}_0}{\rho}\left[\mathbf{a}_\theta\left(\frac{|v-u|^2}{\theta}\right)A\left(\frac{v-u}{\sqrt{\theta}}\right):\nabla_x u+\frac{1}{\sqrt{\theta}}\mathbf{b}_\theta\left(\frac{|v-u|^2}{\theta}\right)B\left(\frac{v-u}{\sqrt{\theta}}\right)\cdot\nabla_x\theta\right].$$

Now, we can plug in f_1 to get the correction terms on the equations S_1^α. For the pressure term,

$$\langle A(v-u)f_1\rangle=-\theta\big\langle\mathbf{a}_\theta(|v|^2)A(v)\otimes A(v)\mathcal{M}(1,0,1)\big\rangle:\nabla_x u.$$

By symmetry, $\langle\mathbf{a}_\theta A_{ij}A_{kl}\mathcal{M}(1,0,1)\rangle$ is not zero only when (i) $i=k\neq j=l$, (ii) $i=l\neq j=k$, (iii) $i=j\neq k=l$, (iv) $i=j=k=l$.

Exercise 12. *Prove the following identity: for $D\geq 2$,*

$$\theta\big\langle\mathbf{a}_\theta(|v|^2)A(v)\otimes A(v)\mathcal{M}(1,0,1)\big\rangle=\mu(\theta)\left(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}-\frac{2}{D}\delta_{ij}\delta_{kl}\right),$$

where $\mu=\mu(\theta)$ is the viscosity coefficient: $\mu(\theta)=\theta\langle\mathbf{a}_\theta(|v|^2)v_1^2v_2^2\mathcal{M}(1,0,1)\rangle$. Moreover, μ is non-negative.

Therefore, the correction on the momentum equation is $\nabla_x\cdot(\mu(\theta)\mathcal{D}(u))$, where

$$\mathcal{D}(u)=\nabla_x u+(\nabla_x u)^T-\frac{2}{D}(\nabla_x\cdot u)\mathbb{I}$$

is the *symmetric deformation tensor*.

A similar argument can be used to compute the correction on the energy equation.

$$\begin{aligned}&-\nabla_x\cdot\langle A(v-u)uf_1\rangle=\nabla_x\cdot(\mu(\theta)\mathcal{D}(u)u),\\&-\nabla_x\cdot\langle B(v-u)f_1\rangle=\partial_{x_j}\big(\theta\langle\mathbf{b}_\theta(|v|^2)B_i(v)B_j(v)\mathcal{M}(1,0,1)\rangle\partial_{x_i}\theta\big).\end{aligned}$$

Again, by symmetry $\langle\mathbf{b}_\theta(|v|^2)B_i(v)B_j(v)\mathcal{M}(1,0,1)\rangle$ is not zero only when $i=j$. So,

$$-\nabla_x\cdot\langle B(v-u)f_1\rangle=\nabla_x\cdot(\kappa(\theta)\nabla_x\theta),$$

where the *heat conduction* κ is given by

$$\kappa(\theta) = \frac{1}{4}\langle \mathbf{b}_\theta [|v|^2 - (D+2)]^2 v_1^2 \mathcal{M}(1,0,1)\rangle.$$

Finally, we reach the first order correction to the compressible Euler equations.

$$\begin{cases} \partial_t \rho + \nabla_x \cdot (\rho u) = 0, \\ \partial_t(\rho u) + \nabla_x \cdot (\rho u \otimes u + \rho\theta \mathbb{I}) = \epsilon \nabla_x \cdot (\mu(\theta)\mathcal{D}(u)), \\ \partial_t \left(\frac{1}{2}\rho|u|^2 + \frac{D}{2}\rho\theta\right) + \nabla_x \cdot \left(\frac{1}{2}\rho|u|^2 u + \frac{D+2}{2}\rho\theta u\right) = \epsilon\nabla_x \cdot (\mu(\theta)\mathcal{D}(u)u + \kappa(\theta)\nabla_x\theta). \end{cases}$$

This corresponds to *Compressible Navier-Stokes equations*.

Note that for hard sphere collision, $\mathbf{a}_\theta = \theta^{-1/2}\mathbf{a}$ and $\mathbf{b}_\theta = \theta^{-1/2}\mathbf{b}$. Hence, $\mu(\theta) = \theta^{1/2}\mu_0$ and $\kappa(\theta) = \theta^{1/2}\kappa_0$, where μ_0 and κ_0 does not depend on θ.

One can compute higher order corrections of compressible Euler equations through Chapman-Enskog expansion. The second order correction is called *Burnett equations*, which is ill-posed. So we shall not investigate higher order corrections.

2.4 Acoustic limit

Consider a small perturbation around a global Maxwellian $\mathcal{M}(1,0,1)$. The initial value problem reads

$$\partial_t f_\epsilon + v\cdot\nabla_x f_\epsilon = \frac{1}{\epsilon}\mathcal{B}[f_\epsilon, f_\epsilon], \quad f_\epsilon^{in} = \mathcal{M}(1+\eta_\epsilon\rho^{in}, \eta_\epsilon u^{in}, 1+\eta_\epsilon\theta^{in}),$$

where $\eta_\epsilon \to 0$ as $\epsilon \to 0$. We denote $\mathcal{M} = \mathcal{M}(1,0,1)$ in this section. Let

$$g_\epsilon = \frac{f_\epsilon - \mathcal{M}}{\eta_\epsilon \mathcal{M}},$$

which describes the perturbation. Express Boltzmann equation in terms of g_ϵ through the following calculation.

$$\begin{aligned} &(\partial_t + v\cdot\nabla_x) f_\epsilon = \eta_\epsilon \mathcal{M}\ (\partial_t + v\cdot\nabla_x) g_\epsilon, \\ &\mathcal{B}[f_\epsilon, f_\epsilon] = \int_{\mathbb{S}^{D-1}\times\mathbb{R}^D} \mathcal{M}\mathcal{M}_*[(1+\eta_\epsilon g_\epsilon')(1+\eta_\epsilon g_{\epsilon*}') - (1+\eta_\epsilon g_\epsilon)(1+\eta_\epsilon g_{\epsilon*})]b\ d\omega dv_* \\ &\qquad = \eta_\epsilon \mathcal{M}\mathcal{L}_\mathcal{M}[g_\epsilon] + \eta_\epsilon^2 \mathcal{M} \underbrace{\int_{\mathbb{S}^{D-1}\times\mathbb{R}^D} (g_\epsilon' g_{\epsilon*}' - g_\epsilon g_{\epsilon*})\mathcal{M}_* b\ d\omega dv_*}_{:=\tilde{\mathcal{B}}_\mathcal{M}[g_\epsilon, g_\epsilon]}. \end{aligned}$$

Therefore, we obtain (up to some small corrections) *linearized Boltzmann equation*

$$\partial_t g_\epsilon + v\cdot\nabla_x g_\epsilon = \frac{1}{\epsilon}\mathcal{L}_\mathcal{M}[g_\epsilon] + \frac{\eta_\epsilon}{\epsilon}\tilde{\mathcal{B}}_\mathcal{M}[g_\epsilon, g_\epsilon].$$

We assume the formal limt $g_\epsilon \to g$, Then the leading order term $\mathcal{L}_\mathcal{M} g = 0$. By Lemma 2.1, we know g is collision invariant. So we can write

$$g(t,x,v) = c_1(t,x) + c_2(t,x)\cdot v + c_3(t,x)|v|^2.$$

Let (ρ, u, θ) be macroscopic quantities such that

$$\rho = \langle g\mathcal{M}\rangle, \quad u = \langle v g\mathcal{M}\rangle, \quad \frac{D}{2}(\rho+\theta) = \left\langle \frac{1}{2}|v|^2 g\mathcal{M}\right\rangle.$$

Note that $(\rho, u, \frac{D}{2}(\rho+\theta))$ is the η_ϵ order fluctuations for $(\rho, \rho u, \mathcal{E})$.

Then, we can express (c_1, c_2, c_3) by (ρ, u, θ) and get

$$g(t,x,v) = \rho(t,x) + u(t,x)\cdot v + \theta(t,x)\left(\frac{|v|^2}{2} - \frac{D}{2}\right). \tag{2.8}$$

Observing that for $\xi_\alpha = \{1, v, \frac{1}{2}|v|^2\}$, one can easily check

$$\langle \xi_\alpha \mathcal{L}_\mathcal{M}[g_\epsilon]\mathcal{M}\rangle = \langle \xi_\alpha \tilde{\mathcal{B}}_\mathcal{M}[g_\epsilon, g_\epsilon]\mathcal{M}\rangle = 0.$$

Hence, we have the following local conservation laws

$$\begin{cases} \partial_t\langle g\mathcal{M}\rangle + \nabla_x\cdot\langle v g\mathcal{M}\rangle = 0 \\ \partial_t\langle v g\mathcal{M}\rangle + \nabla_x\cdot\langle v\otimes v g\mathcal{M}\rangle = 0 \\ \partial_t\langle \frac{1}{2}|v|^2 g\mathcal{M}\rangle + \nabla_x\cdot\langle \frac{1}{2}v|v|^2 g\mathcal{M}\rangle = 0. \end{cases}$$

Plug in the ansatz (2.8) to close the system. We reach the *acoustic system*

$$\begin{cases} \partial_t\rho + \nabla_x\cdot u = 0 \\ \partial_t u + \nabla_x(\rho+\theta) = 0 \\ \frac{D}{2}\partial_t\theta + \nabla_x\cdot u = 0, \end{cases} \quad \text{or} \quad \partial_t\begin{bmatrix}\rho\\ u\\ \theta\end{bmatrix} + \begin{bmatrix} 0 & \nabla_x\cdot & 0 \\ -\nabla_x & 0 & -\nabla_x \\ 0 & \frac{2}{D}\nabla_x\cdot & 0\end{bmatrix}\begin{bmatrix}\rho\\ u\\ \theta\end{bmatrix} = 0.$$

It is a linear first order system of conservation laws. Since the differential operator is skew adjoint with respect to L^2 inner product, one can apply Hille-Yoshida theory to obtain global wellposedness of the system.

The acoustic limit is the first example where Ma$\ll$St. Indeed, the scale for macroscopic velocity $u_A = \mathcal{O}(\eta_\epsilon) \ll 1$. So Ma$\ll$1=St. We will see other examples later in the notes.

2.5 Incompressible limits

One interesting regime is the long time behavior of the perturbative system. This corresponds to the case where the time scale $\tau_A \gg 1$, and St$\ll 1$. Take St$=\delta_\epsilon$. The dynamics on g_ϵ reads

$$\delta_\epsilon\partial_t g_\epsilon + v\cdot\nabla_x g_\epsilon = \frac{1}{\epsilon}\mathcal{L}_\mathcal{M}[g_\epsilon] + \frac{\eta_\epsilon}{\epsilon}\tilde{\mathcal{B}}_\mathcal{M}[g_\epsilon, g_\epsilon].$$

When $\delta_\epsilon = 1$, we obtain the acoustic limit. Now, let us consider $\delta_\epsilon \ll 1$. The leading order term of the system is still $\mathcal{L}_\mathcal{M}[g_\epsilon] = 0$, which formally leads to the limiting profile (2.8). The macroscopic quantities (ρ, u, θ) satisfy local conservation laws

$$\begin{cases} \nabla_x \cdot \langle v g \mathcal{M} \rangle = 0 \\ \nabla_x \cdot \langle v \otimes v g \mathcal{M} \rangle = 0 \\ \nabla_x \cdot \langle \frac{1}{2} v |v|^2 g \mathcal{M} \rangle = 0. \end{cases}$$

Plug in the Ansatz (2.8) and the leading order terms become

$$\nabla_x \cdot u = 0, \quad \nabla_x(\rho + \theta) = 0, \quad \nabla_x \cdot u = 0.$$

The divergence free condition on the macroscopic velocity u is called *incompressibility condition*. The equality on (ρ, θ) implies the *Boussinesq relation* $\rho + \theta = 0$.

Since the leading order terms do not provide full information to the macroscopic quantities, we need to study higher order terms.

$$\begin{cases} \delta_\epsilon \partial_t \langle g_\epsilon) \mathcal{M} \rangle + \nabla_x \cdot \langle v g_\epsilon \mathcal{M} \rangle = 0 \\ \delta_\epsilon \partial_t \langle v g_\epsilon \mathcal{M} \rangle + \nabla_x \cdot \langle v \otimes v g_\epsilon \mathcal{M} \rangle = 0 \\ \delta_\epsilon \partial_t \langle \frac{1}{2} |v|^2 g_\epsilon \mathcal{M} \rangle + \nabla_x \cdot \langle \frac{1}{2} v |v|^2 g_\epsilon \mathcal{M} \rangle = 0. \end{cases}$$

Let us focus on the momentum equation.

$$\partial_t u_\epsilon = -\frac{1}{\delta_\epsilon} \nabla_x \left\langle \frac{1}{D} |v|^2 g_\epsilon \mathcal{M} \right\rangle - \frac{1}{\delta_\epsilon} \nabla_x \cdot \langle A(v) g_\epsilon \mathcal{M} \rangle.$$

For the first term, we know $\lim_{\epsilon \to 0} \left\langle \frac{1}{D} |v|^2 g_\epsilon \mathcal{M} \right\rangle = \rho + \theta = 0$, using Boussinesq relation. However, it is generally not known whether the limit $\lim_{\epsilon \to 0} (\rho_\epsilon + \theta_\epsilon)/\delta_\epsilon$ exists. What we do know is that it is a gradient term. For the second term, we apply Lemma 2.2 and get

$$\begin{aligned} \partial_t u_\epsilon &= -\frac{1}{\delta_\epsilon} \nabla_x \cdot \big(\mathcal{L}_\mathcal{M}[-\mathbf{a}(|v|^2) A(v)], g_\epsilon \big)_\mathcal{M} = \frac{1}{\delta_\epsilon} \nabla_x \cdot \big(\mathbf{a}(|v|^2) A(v), \mathcal{L}_M[g_\epsilon] \big)_\mathcal{M} \\ &= \epsilon \underbrace{\nabla_x \cdot \big(\mathbf{a}(|v|^2) A(v), \partial_t g_\epsilon \big)_\mathcal{M}}_{I} + \frac{\epsilon}{\delta_\epsilon} \underbrace{\nabla_x \cdot \big(\mathbf{a}(|v|^2) A(v), v \cdot \nabla_x g_\epsilon \big)_\mathcal{M}}_{II} \\ &\quad - \frac{\eta_\epsilon}{\delta_\epsilon} \underbrace{\nabla_x \cdot \big(\mathbf{a}(|v|^2) A(v), \tilde{\mathcal{B}}_\mathcal{M}[g_\epsilon, g_\epsilon] \big)_\mathcal{M}}_{III}, \qquad \text{modulo a gradient term.} \end{aligned}$$

One scaling factor that distinguishes different incompressible limits is the Reynolds number $\mathrm{Re} = \frac{\mathrm{Ma}}{\mathrm{Kn}} = \frac{\eta_\epsilon}{\epsilon}$. We will discuss the following three regimes.

- Re$\to 0$, St=Kn: incompressible Stokes limit,
- Re$\in (0, +\infty)$, St=Kn: incompressible Navier-Stokes limit,
- Re$\to +\infty$, St=Ma: incompressible Euler limit.

2.5.1 Stokes limit

Take $\delta_\epsilon = \epsilon$. This corresponds to the case where Kn and St has the same order. Equivalently, it means that the Reynolds number is finite. We also take the size of the fluctuation $\eta_\epsilon = o(\epsilon)$. Under this setup, the leading order term for the dynamics of u_ϵ is $\partial_t u_\epsilon = II$.

When $\epsilon \to 0$, plug in the Ansatz (2.8) and get

$$II_i = \partial_{x_j}\partial_{x_k}\big\langle \mathbf{a}(|v|^2)A_{ij}(v)v_k g\mathcal{M}\big\rangle = \big\langle \mathbf{a}(|v|^2)A_{ij}(v)v_k v_l\mathcal{M}\big\rangle \partial_{x_j}\partial_{x_k}u_l.$$

Similar as Exercise 12, one can check the following identity

$$\big\langle \mathbf{a}(|v|^2)A_{ij}(v)v_k v_l\mathcal{M}\big\rangle = \mu\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \frac{2}{D}\delta_{ij}\delta_{kl}\right),$$

where the viscosity coefficient $\mu = \langle \mathbf{a}(|v|^2)v_1^2 v_2^2\mathcal{M}\rangle > 0$.

So, the limiting dynamics on u becomes

$$\partial_t u = \mu\left[\left(1 - \frac{2}{D}\right)\nabla_x(\nabla_x\cdot u) + \Delta_x u\right].$$

Since u is incompressible, we end up with the *Stokes limit*

$$\partial_t u + \nabla_x p = \mu\Delta_x u, \quad \nabla_x\cdot u = 0.$$

Note that the presure p should satisfy $\Delta_x p = 0$ due to incompressibility. Since we do not take into account boundary effect, $p = 0$.

2.5.2 Navier-Stokes limit

The Navier-Stokes limit corresponds the case where $\delta_\epsilon = \eta_\epsilon = \epsilon$. If this case where fluctuation has the same size as Knudsen number, the leading order term for the dynamics of u_ϵ is $\partial_t u_\epsilon = II - III$.

The following lemma is useful to find the limit for term III.

Lemma 2.3. *If $g \in \mathrm{Ker}\mathcal{L}_\mathcal{M}$, then $\tilde{\mathcal{B}}[g,g] = -\frac{1}{2}\mathcal{L}_\mathcal{M}(g^2)$.*

Proof. From Lemma 2.1, we know if $g \in \mathrm{Ker}\mathcal{L}_\mathcal{M}$, then $g' + g'_* = g + g_*$. Compute

$$\begin{aligned}\mathcal{L}_\mathcal{M}[g^2] &= \int_{\mathbb{S}^{D-1}\times\mathbb{R}^D}\big((g')^2 + (g'_*)^2 - g^2 - g_*^2\big)\mathcal{M}_* b\, d\omega dv_* \\ &= \int_{\mathbb{S}^{D-1}\times\mathbb{R}^D}\big[\underbrace{(g'+g'_*)^2 - (g+g_*)^2}_{=0} - 2(g'g'_* - gg_*)\big]\mathcal{M}_* b\, d\omega dv_* = -2\tilde{\mathcal{B}}[g].\end{aligned}$$

This concludes the proof. □

We apply the lemma and calculate the limit of III:

$$III = -\frac{1}{2}\nabla_x \cdot \big(\mathbf{a}(|v|^2)A(v), \mathcal{L}_{\mathcal{M}}[g^2]\big)_{\mathcal{M}} = -\frac{1}{2}\nabla_x \cdot \big(A(v), g^2\big)_{\mathcal{M}}$$

Plug in $g = u \cdot v + \theta(\frac{|v|^2}{2} - \frac{D+2}{2})$ and take out all terms with odd symmetry, we get

$$\langle A_{ij}(v)g^2\mathcal{M}\rangle = \langle A_{ij}(v)(u\cdot v)^2\mathcal{M}\rangle + \delta_{ij}\frac{\theta^2}{4}\left\langle \left(v_i^2 - \frac{1}{D}|v|^2\right)(|v|^2-(D+2))^2\mathcal{M}\right\rangle.$$

The second term is clearly zero as A is trace-free. For the first term,

$$\langle A_{ij}(v)v_kv_l\mathcal{M}\rangle u_ku_l = \left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \frac{2}{D}\delta_{ij}\delta_{kl}\right)u_ku_l = 2u_iu_j - \frac{2}{D}\delta_{ij}|u|^2$$

Therefore,

$$III = -\nabla_x \cdot \left(u\otimes u - \frac{1}{D}|u|^2\mathbb{I}\right) = -u\cdot\nabla_x u - u(\nabla_x\cdot u) + \frac{1}{D}\nabla_x(|u|^2).$$

Finally, we derive the *incompressible Navier-Stokes equation*

$$\partial_t u + u\cdot\nabla_x u + \nabla_x p = \mu\Delta_x u, \quad \nabla_x\cdot u = 0.$$

2.5.3 Euler limit

The incompressible Euler limit describes the scenario where

$$\eta_\epsilon = \delta_\epsilon, \quad \lim_{\epsilon\to 0}\eta_\epsilon = +\infty.$$

In this case, the leading order term for the dynamics of u_ϵ is $\partial_t u_\epsilon = -III$. From the above calculation, we can formally obtain the *incompressible Euler equation*:

$$\partial_t u + u\cdot\nabla_x u + \nabla_x p = 0, \quad \nabla_x\cdot u = 0.$$

2.5.4 Temperature dynamics

To derive the dynamics of θ, we consider the conservation law

$$\partial_t\left\langle\frac{1}{2}(|v|^2-(D+2))g_\epsilon\mathcal{M}\right\rangle + \frac{1}{\delta_\epsilon}\nabla_x\cdot\langle\underbrace{\frac{1}{2}(|v|^2-(D+2))v}_{=B(v)}g_\epsilon\mathcal{M}\rangle = 0.$$

Plug in the Ansatz $g = u\cdot v + \theta(\frac{|v|^2}{2} - \frac{D+2}{2})$. The first term becomes

$$\partial_t\left\langle\frac{1}{2}(|v|^2-(D+2))g\mathcal{M}\right\rangle = \left\langle\frac{1}{4}(|v|^2-(D+2))^2\mathcal{M}\right\rangle\partial_t\theta = \frac{D+2}{2}\partial_t\theta,$$

We use a similar argument to treat the second term.

$$
\begin{aligned}
-\frac{1}{\delta_\epsilon}\nabla_x\cdot\langle B(v)g_\epsilon\mathcal{M}\rangle &= \frac{1}{\delta_\epsilon}\nabla_x\cdot\big(\mathcal{L}_\mathcal{M}[\mathbf{b}(|v|^2)B(v)],g_\epsilon\big)_\mathcal{M} = \frac{1}{\delta_\epsilon}\nabla_x\cdot\big(\mathbf{b}(|v|^2)B(v),\mathcal{L}_M[g_\epsilon]\big)_\mathcal{M} \\
&= \ \epsilon\underbrace{\nabla_x\cdot\big(\mathbf{a}(|v|^2)A(v),\partial_t g_\epsilon\big)_\mathcal{M}}_{I} + \frac{\epsilon}{\delta_\epsilon}\underbrace{\nabla_x\cdot\big(\mathbf{b}(|v|^2)B(v),v\cdot\nabla_x g_\epsilon\big)_\mathcal{M}}_{II} \\
&\quad -\frac{\eta_\epsilon}{\delta_\epsilon}\underbrace{\nabla_x\cdot\big(\mathbf{b}(|v|^2)B(v),\tilde{\mathcal{B}}_\mathcal{M}[g_\epsilon,g_\epsilon]\big)_\mathcal{M}}_{III}.
\end{aligned}
$$

Term II represemts the heat diffusion. The formal limit can be computed as

$$
II = \frac{1}{4}\langle\mathbf{b}(|v|^2)(|v|^2-(D+2))^2 v_i v_j\mathcal{M}\rangle\partial_{x_i}\partial_{x_j}\theta = \kappa\Delta\theta,
$$

where the heat conduction $\kappa = \frac{1}{4}\langle\mathbf{b}(|v|^2)(|v|^2-(D+2))^2 v_1^2\mathcal{M}\rangle > 0$.

Term III represents the heat convection. By Lemma 2.3,

$$
\begin{aligned}
III &= -\frac{1}{2}\nabla_x\cdot\big(\mathbf{b}(|v|^2)B(v),\mathcal{L}_\mathcal{M}[g^2]\big)_\mathcal{M} = -\frac{1}{2}\nabla_x\cdot\big(B(v),g^2\big)_\mathcal{M} \\
&= -\frac{1}{4}\langle(|v|^2-(D+2))^2 v_i v_j\mathcal{M}\rangle\partial_{x_i}(u_i\theta) = -\nabla_x\cdot(u\theta) = -u\cdot\nabla_x\theta.
\end{aligned}
$$

Therefore, in the Stokes limit, temperature fluctuation satisfies *heat equation*

$$
\partial_t\theta = \kappa\Delta\theta.
$$

In the Navier-Stokes limit, it satisfies *Fourier equation*

$$
\partial_t\theta + u\cdot\nabla_x\theta = \kappa\Delta\theta.
$$

In the Euler limit, it satisfies the *transport equation*

$$
\partial_t\theta + u\cdot\nabla_x\theta = 0.
$$

Chapter 3

Global wellposedness theory for Boltzmann equation

3.1 Ukai's theory on perturbations around Maxwellian

Consider the following Cauchy problem for Boltzmann equation

$$\partial_t f + v \cdot \nabla_x f = \mathcal{B}[f, f],$$

with initial profile close to a Maxwellian distribution

$$f^{in} = \mathcal{M}(1 + g^{in}).$$

Without loss of generality, we scale f so that

$$\int_\Omega \left\langle \begin{bmatrix} 1 \\ v \\ \frac{|v|^2}{2} \end{bmatrix} f \right\rangle dx = \begin{bmatrix} 1 \\ 0 \\ \frac{D}{2}, \end{bmatrix}$$

and take $\mathcal{M} = \mathcal{M}(1, 0, 1)$ be the standard Gaussian. To avoid boundary condition, take $\Omega = \mathbb{T}^D$. The goal is to obtain global existence of classical solution for small initial data g^{in}.

We can write the equation with respect to the fluctuation g:

$$\partial_t g + v \cdot \nabla_x g = \mathcal{L}_{\mathcal{M}}[g] + \tilde{\mathcal{B}}_{\mathcal{M}}[g, g].$$

Clearly, $\int \langle \xi_\alpha g \mathcal{M} \rangle dx = 0$. Let $\mathcal{P}$ be the projection onto the nullspace $\mathcal{N}$ of $\mathcal{L}_{\mathcal{M}}$ under inner product $(\cdot, \cdot)_{\mathcal{M}}$, namely

$$\mathcal{P} g = \sum_{\alpha=1}^{D+2} \frac{(\xi_\alpha, g)_{\mathcal{M}}}{(\xi_\alpha, \xi_\alpha)_{\mathcal{M}}} \xi_\alpha.$$

Then, we have the relation

$$\int_{\mathbb{T}^D} \mathcal{P} g dx = \sum_{\alpha=1}^{D+2} \frac{\xi_\alpha}{(\xi_\alpha, \xi_\alpha)_{\mathcal{M}}} \int_{\mathbb{T}^D} (\xi_\alpha, g)_{\mathcal{M}} dx = 0, \quad \forall\, \xi \in \mathbb{R}^D. \tag{3.1}$$

As we will see later, the linear operator $-v \cdot \nabla_x + \mathcal{L}_{\mathcal{M}}$ generates a $\mathcal{C}^0$-semigroup S. Therefore, we can write the solution of g using Duhamel's principle.

$$g(t) = S(t) g^{in} + \int_0^t S(t-s) \tilde{\mathcal{B}}_{\mathcal{M}}[g, g] ds.$$

Let

$$\Phi[g](t) = S(t) g^{in} + \int_0^t S(t-s) \tilde{\mathcal{B}}_{\mathcal{M}}[g, g] ds.$$

Then, the solution g satisfies $\Phi[g] = g$, that is, the solution is a fixed point of the map Φ. The plan is to show Φ is a contraction map in an appropriate space, if g^{in} is small enough.

The functional space where the fixed point argument will be applied is

$$H_{l,k} := \big\{ g = g(x,v) \;\big|\; \|g\|_{l,k} = \sup_v (1+|v|)^k \|\mathcal{M}^{1/2} g(\cdot, v)\|_{H^l_x} < +\infty \big\}.$$

3.1.1 Linearized Boltzmann operator

In this part, we study the operator $-v\cdot\nabla_x + \mathcal{L}_{\mathcal{M}}$ and the spectral analysis of the semigroup generated by the operator.

Let us quickly recall the classical theory on semigroups. Denote X be a Banach space with norm $\|\cdot\|$. We call $\{S(t)\}_{t\geq 0}$ a *strongly continuous semigroup* (or $\mathcal{C}^0$-semigroup) if:

(a). $S(t)$ is a linear bounded operator on X, for all $t \geq 0$.

(b). $S(0) = \mathbb{I}$, and $S(t+s) = S(t)S(s)$ for $t, s \geq 0$.

(c). $\lim_{t\to 0} S(t) = \mathbb{I}$ strongly in X.

Clearly, if $g^{in} \in X$, then $S(t) g^{in} \in \mathcal{C}^0([0,\infty); X)$.

Take the Hilbert decomposition of the linearized Boltzmann operator

$$\mathcal{L}_{\mathcal{M}}[g] = -\nu g + \mathcal{K}[g],$$

where ν and $\mathcal{K}$ are defined in (2.7).

Let us assume that the collision cross section has the form $b(|v-v_*|, \omega) = |v-v_*|^\gamma h(\omega)$, where $\gamma \in [0,1]$ and $h \in L^1(\mathbb{S}^{D-1})$. Note that the assumption holds for the hard sphere case with $\gamma = 1$.

Exercise 13. *Prove that the collision frequencey ν is radial, and there exists $c_0, c_1 > 0$ such that*

$$c_0|v|^\gamma \le \nu(v) \le c_1(1+|v|)^\gamma.$$

So, ν is unbounded. Another importatnt observation is $\nu(|v|) \ge \nu_0 > 0$. Indeed, we compute

$$\begin{aligned}\nu'(r) =& \frac{v}{|v|}\cdot\nabla_v \int_{\mathbb{S}^{D-1}\times\mathbb{R}^D} \mathcal{M}_* b \, d\omega dv_* \\ =& \underbrace{\frac{\gamma}{(2\pi)^{D/2}|v|}\int_{\mathbb{S}^{D-1}} h(\omega)d\omega}_{>0} \int_{\mathbb{R}^D} e^{-\frac{|v_*|^2}{2}}|v-v_*|^{\gamma-2}(v-v_*)\cdot v \, dv_*.\end{aligned}$$

For the last integral, write $|v_*|^2 = |v|^2 + |v-v_*|^2 - 2(v-v_*)\cdot v$. Due to odd symmetry, only the third part provides nonzero contribution to the integral, and clearly $\nu'(r) > 0$. Therefore, $\nu(r) \ge \nu(0) > 0$, and ν satisfies

$$C_0(1+|v|)^\gamma \le \nu(v) \le C_1(1+|v|)^\gamma. \tag{3.2}$$

The positive lower bound on ν provides contraction to the semigroup with generator $\mathsf{A} = -v\cdot\nabla_x - \nu$. To see this, we solve the equation

$$\partial_t g = \mathsf{A}[g] = -v\cdot\nabla_x g - \nu g$$

explicitly, and get the solution

$$g(t,x,v) = e^{t\mathsf{A}} g^{in} = e^{-\nu(v)t} g^{in}(x - vt, v).$$

Lemma 3.1. *$e^{t\mathsf{A}}$ is a $\mathcal{C}^0$-semigroup on $X = L^2_{\mathcal{M}}(\mathbb{R}^D; L^2(\mathbb{T}^D))$ with*

$$\mathrm{Dom}(\mathsf{A}) = \{g \in X \mid v\cdot\nabla_x g,\ \nu g \in X\}.$$

Moreover, $e^{t\mathsf{A}}$ is a contraction semigroup with $\|e^{t\mathsf{A}}\| \le e^{-\nu_0 t}$.

Proof. From the explicit solution, it is easy to check $\|g(t)\| \le e^{-\nu_0 t}\|g^{in}\|$. It remains to check $e^{t\mathsf{A}}$ is a semigroup. (a) and (b) are trivial. For (c), take the Fourier series of g in x. We get

$$\hat{g}(t,k,v) = e^{-(ik\cdot v+\nu(v))t}\hat{g}^{in}(k,v), \quad k \in \mathbb{Z}^D.$$

Note that for $z \in \mathbb{C}$ with $\mathrm{Re}(z) < 0$, one has $|e^{tz} - 1| \to 0$ as $t \to 0$, and $|e^{tz} - 1| \le 2$ for $t \ge 0$. Taking $z = -(ik\cdot v + \nu(v))t$ and applying Lebesgue dominated convergence theorem (DCT), we get

$$\lim_{t\to 0}\|\hat{g}(t) - \hat{g}^{in}\|_{(L^2_{\mathcal{M}})_v l^2_k} = 0.$$

Therefore, (c) holds by Parseval's identity.

To check A is the generator of $e^{t\mathsf{A}}$, we need to prove

$$\lim_{t\to 0+}\left\|\frac{e^{t\mathsf{A}}-\mathbb{I}}{t}-\mathsf{A}\right\|=0.$$

It follows from a similar argument using DCT on the Fourier side. The domain $\mathrm{Dom}(\mathsf{A})$ is chosen so that $e^{-(ik\cdot v+\nu(v))}\hat{g}^{in}(x,k)\in(L^2_{\mathcal{M}})_v l^2_k$ and DCT can be applied. □

Now, we consider the linearized Boltzmann equation. It can be written as $\partial_t g=\mathsf{B}[g]$ where

$$\mathsf{B}[g]=-v\cdot\nabla_x g+\mathcal{L}_{\mathcal{M}}[g]=\mathsf{A}[g]+\mathcal{K}[g].$$

Since $\mathcal{K}$ is a bounded linear operator from $L^2_{\mathcal{M}}$ to $L^2_{\mathcal{M}}$, by standard perturbation theory on semigroup, we know $e^{t\mathsf{B}}$ is semigroup in X with generator B. Moreover, we can write the solution by Duhamel's principle

$$\|e^{t\mathsf{B}}g^{in}\|\le\|e^{t\mathsf{A}}g^{in}\|+\|\mathcal{K}\|\int_0^t\|e^{(t-s)\mathsf{A}}\|\|e^{s\mathsf{B}}g^{in}\|ds\le e^{-\nu_0 t}\|g^{in}\|+\|\mathcal{K}\|\int_0^t e^{-\nu(t-s)}\|e^{s\mathsf{B}}g^{in}\|ds$$

and obtain the following rough estimate using Gronwall's inequality

$$\|e^{t\mathsf{B}}\|\le e^{(-\nu_0+\|\mathcal{K}\|)t}.$$

Such estimate does not imply that $e^{t\mathsf{B}}$ is a contraction semigroup, as $-\nu_0+\|\mathcal{K}\|$ is not necessarily negative. Therefore, we need a stronger estimate.

Theorem 3.2. *Let $X=L^2_{\mathcal{M}}(\mathbb{R}^D;L^2(\mathbb{T}^D))$. Then,*

$$\|e^{t\mathsf{B}}g^{in}\|\le C\|g^{in}\|,$$

where the constant C is independent of t and g^{in}. Moreover, if g^{in} satisfies the relation (3.1)*, then there exists $\sigma>0$, such that*

$$\|e^{t\mathsf{B}}g^{in}\|\le Ce^{-\sigma t}\|g^{in}\|.$$

Proof. Let $\hat{\mathsf{A}}(k),\hat{\mathsf{B}}(k)$ be the Fourier multiplier corresponding to $\mathsf{A}(t,\cdot,v),\mathsf{B}(t,\cdot,v)$ respectively,

$$\hat{\mathsf{A}}(k)=-iv\cdot k-\nu(v),\quad \hat{\mathsf{B}}(k)=-iv\cdot k+\mathcal{L}_{\mathcal{M}}.$$

Since $e^{t\mathsf{A}}$ is a contraction semigroup, we know that the resolvent

$$\rho(\hat{\mathsf{A}}(k))\supset\{\lambda\in\mathbb{C}\mid \mathrm{Re}\lambda>-\nu_0\}.$$

As $\mathcal{K}$ is a compact operator from $L^2_{\mathcal{M}}$ to $L^2_{\mathcal{M}}$, then $\hat{\mathsf{B}}(k)$ is a compact perturbation of $\hat{\mathsf{A}}(k)$. So it has only discrete point spectrum in $\{\lambda\in\mathbb{C}\mid\mathrm{Re}\lambda>-\nu_0+\delta\}$, for any $\delta>0$.

Let λ be a point spectrum (i.e. eigenvalue) of $\hat{\mathsf{B}}(k)$, with its eigenfunction φ. Compute

$$0 = \mathrm{Re}\big((\lambda - \hat{\mathsf{B}}(k))\varphi, \varphi\big)_{\mathcal{M}} = \mathrm{Re}\lambda \|\varphi\|^2_{\mathcal{M}} - (\mathcal{L}_{\mathcal{M}}\varphi, \varphi)_{\mathcal{M}} \geq \mathrm{Re}\lambda \|\varphi\|^2_{\mathcal{M}}.$$

It implies $\mathrm{Re}\lambda \leq 0$. So, we obtain that $\|e^{t\hat{\mathsf{B}}(k)}\| \leq C$ and hence

$$\|e^{t\mathsf{B}} g^{in}\| = \|e^{t\hat{\mathsf{B}}} \hat{g}^{in}\|_{(L^2_{\mathcal{M}})_v l^2_k} \leq C \|\hat{g}^{in}\|_{(L^2_{\mathcal{M}})_v l^2_k} = C\|g^{in}\|.$$

To obtain the decay estimate, we need to exclude $\mathrm{Re}\lambda = 0$. Suppose so, then we must have $(\mathcal{L}_{\mathcal{M}}\varphi, \varphi)_{\mathcal{M}} = 0$. By Lemma 2.1, $\varphi \in \mathcal{N}$. Therefore,

$$(\lambda - \hat{\mathsf{B}}(k))\varphi = (\lambda + iv \cdot k)\varphi.$$

The term is zero if and only if $k = 0$ and $\mathrm{Im}\lambda = 0$. It shows that when $k \neq 0$, there is no eigenvalue with zero real part. Therefore, given $k \neq 0$, there exists a $\sigma(k) > 0$ such that $\rho(\hat{\mathsf{B}}(k)) \supset \{\lambda \in \mathbb{C} \mid \mathrm{Re}\lambda > -\sigma(k)\}$, and equivalently $\|e^{t\hat{\mathsf{B}}(k)}\| \leq Ce^{-\sigma(k)t}$.

Now we show that there exists $\sigma > 0$ such that $\sigma \leq \sigma(k)$ for all $k \in \mathbb{Z}^D \backslash \{0\}$.

Take $\lambda \in \{\lambda \in \mathbb{C} \ : \ \mathrm{Re}\lambda \in (-\nu_0, 0)\} \subset \rho(\hat{\mathsf{A}}(k))$. Then,

$$(\lambda - \hat{\mathsf{B}}(k))^{-1} = (\lambda - \hat{\mathsf{A}}(k))^{-1}\big(\mathbb{I} - \mathcal{K}(\lambda - \hat{\mathsf{A}}(k))^{-1}\big)^{-1}.$$

The kernel $\mathcal{K}$ has the following property

$$\sup_{\mathrm{Re}\lambda \geq -\nu_0 + \delta} \|\mathcal{K}(\lambda - \hat{\mathsf{A}}(k))^{-1}\| \leq C(\delta, D)(1 + |k|)^{-\gamma(D)},$$

where $C(\delta, D)$ and $\gamma(D)$ are positive constants. We omit the proof here. So, if $|k|$ is sufficiently large, namely $|k| > k_0(\delta, D)$, then $\|\mathcal{K}(\lambda - \hat{\mathsf{A}}(k))^{-1}\| < 1/2$, and so $\lambda \in \rho(\hat{\mathsf{B}}(k))$. Therefore, $\sigma(k) \geq \nu - \delta$ for all $|k| > k_0(\delta, D)$ and we find

$$\sigma = \min\left\{ \min_{1 \leq |k| \leq k_0} \sigma(k), \nu - \delta \right\} > 0.$$

Note that relation (3.1) implies that $\hat{g}(t, 0, v) \equiv 0$. Therefore,

$$\|e^{t\mathsf{B}} g^{in}\| = \|e^{t\hat{\mathsf{B}}} \hat{g}^{in} \mathbb{1}_{k \neq 0}\|_{(L^2_{\mathcal{M}})_v l^2_k} \leq Ce^{-\sigma t} \|\hat{g}^{in}\|_{(L^2_{\mathcal{M}})_v l^2_k} = Ce^{-\sigma t}\|g^{in}\|.$$

□

One can extend the thorem to other spaces like $L^2_{\mathcal{M}}(\mathbb{R}^D; H^l(\mathbb{T}^D))$ or $H_{l,k}$. There is also a version of the theorem for $\Omega = \mathbb{R}^D$. In this case, the fourier variable $k \in \mathbb{R}^D$. It is delicate to control $\sigma(k)$ when k is close to zero. The estimate will only provide polynomial decay in time.

3.1.2 Global existence

To get global existence, we need an estimate on the quadratic term

$$\left\| \int_0^t e^{(t-s)\mathsf{B}} \tilde{\mathcal{B}}[g_1, g_2](s) ds \right\| \leq C \|g_1\| \|g_2\|, \tag{3.3}$$

with appropriate choice of the norm $\|\cdot\|$. Given (3.3), we can run a fixed point argument on $\{g \ : \ \|g\| \leq R\}$. Indeed, one has

$$\|\Phi[g]\| \leq \|g^{in}\| + C\|g\|^2 \leq \|g^{in}\| + CR^2.$$

If we pick g^{in} such that $\|g^{in}\| \leq R - CR^2$, then $\|\Phi[g]\| \leq R$. Note that $R - CR^2 > 0$ if R is sufficiently small. For the contraction property, we use the fact that $\tilde{\mathcal{B}}$ is bilinear, namely

$$\tilde{\mathcal{B}}[g_1, g_1] - \tilde{\mathcal{B}}[g_2, g_2] = \tilde{\mathcal{B}}[g_1 + g_2, g_1 - g_2].$$

Therefore, we get

$$\|\Phi[g_1](t) - \Phi[g_2](t)\| \leq C\|g_1 + g_2\| \|g_1 - g_2\| \leq 2CR\|g_1 - g_2\|,$$

which leads to a contraction for $R < (2C)^{-1}$.

The rest of the section is devoted to find a normed space such that (3.3) is satisfied.

First, we estimate the quadratic term $\tilde{\mathcal{B}}_{\mathcal{M}}[g, g]$.

Lemma 3.3. *Let $p \in [1, \infty]$. There exists a constant $C > 0$ such that for all $g_1, g_2 \in L^p_{\mathcal{M}}$, it holds that*

$$\|\nu^{-1} \tilde{\mathcal{B}}_{\mathcal{M}}[g_1, g_2]\|_{L^p_{\mathcal{M}}} \leq C \|g_1\|_{L^p_{\mathcal{M}}} \|g_2\|_{L^p_{\mathcal{M}}}. \tag{3.4}$$

Proof.

$$\tilde{\mathcal{B}}_{\mathcal{M}}[g_1, g_2] = \frac{1}{2} \int_{\mathbb{S}^{D-1} \times \mathbb{R}^D} (g_1' g_{2*}' + g_{1*}' g_2' - g_1 g_{2*} - g_{1*} g_2) \mathcal{M}_* b \, d\omega dv_*.$$

We only estimate one term, and the others follow in the same argument.

$$\int_{\mathbb{S}^{D-1} \times \mathbb{R}^D} g_1' g_{2*}' \mathcal{M}_* b \, d\omega dv_* = \|h\|_{L^1(\mathbb{S}^{D-1})} \|g_1' g_{2*}'\|_{L^p(\mathcal{M}_* dv_*)} \||v - v_*|^\gamma\|_{L^q(\mathcal{M}_* dv_*)}.$$

Note that

$$\||v - v_*|^\gamma\|^q_{L^q(\mathcal{M}_* dv_*)} = \int_{\mathbb{R}^D} |v - v_*|^{q\gamma} \mathcal{M}_* dv_* \leq C(1 + |v|)^{q\gamma} \leq C' \nu(v)^q.$$

Therefore,

$$\nu^{-1} \int_{\mathbb{S}^{D-1} \times \mathbb{R}^D} g_1' g_{2*}' \mathcal{M}_* b \, d\omega dv_* \leq C \|g_1' g_{2*}'\|_{L^p(\mathcal{M}_* dv_*)} = C \left[\int_{\mathbb{R}^D} g_1(v')^p g_2(v_*')^p \mathcal{M}(v_*) dv_* \right]^{1/p}.$$

Take the $L^p_{\mathcal{M}}$ norm with respect to v of the left hand side, we obtain

$$\begin{aligned}\|\text{LHS}\|_{L^p_{\mathcal{M}}} \le & C\left[\int_{\mathbb{R}^D\times\mathbb{R}^D} g_1(v')^p g_2(v'_*)^p \mathcal{M}(v)\mathcal{M}(v_*)dvdv_*\right]^{1/p}\\ =& C\left[\int_{\mathbb{R}^D\times\mathbb{R}^D} g_1(v')^p g_2(v'_*)^p \mathcal{M}(v')\mathcal{M}(v'_*)dv'dv'_*\right]^{1/p} = C\|g_1\|_{L^p_{\mathcal{M}}}\|g_2\|_{L^p_{\mathcal{M}}}.\end{aligned}$$

□

Exercise 14. *Prove that* (3.4) *holds if we change the space* $L^p_{\mathcal{M}}$ *to* $H_{l,k}$.

Let us pretend that ν is bounded. Then, take $X = L^2_{\mathcal{M}}(\mathbb{R}^D; L^2(\mathbb{T}^D))$. Then,

$$\left\|\int_0^t e^{(t-s)\mathsf{B}}\tilde{\mathcal{B}}[g_1,g_2](s)ds\right\| \le \int_0^t e^{-(t-s)\sigma}\|\nu\|_{L^\infty}\|\nu^{-1}\tilde{\mathcal{B}}[g_1,g_2](s)\|ds \le C\sup_{0\le s\le t}\|g_1\|\sup_{0\le s\le t}\|g_2\|.$$

We obtain global existence.

However, from (3.2), we know that ν is unbounded if $\gamma > 0$. Therefore, we need a refined estimate. Define

$$\psi[h] = \int_0^t e^{(t-s)\mathsf{B}}\nu h(s)ds.$$

By Lemma 3.3, it suffies to prove $\|\psi[h]\| \le \|h\|$. Using Duhamel's formula to write

$$\psi[h] = \int_0^t e^{(t-s)\mathsf{A}}\nu h(s)ds + \int_0^t e^{(t-s)\mathsf{A}}\mathcal{K}\psi[h](s)ds = I + II.$$

For the first term, fix v and apply Lemma 3.1 on $H^l(\mathbb{T}^D)$. We get

$$\|I\|_{H^l}(t,v) \le \int_0^t e^{-(t-s)\nu(v)}\nu(v)\|h\|_{H^l}(s,v)ds \le \sup_{0\le s\le t}\|h\|_{H^l}(s,v)ds,$$

where we use the fact that $\int_0^t e^{-(t-s)\nu}\nu ds \le 1$. Multiply both side by $(1+|v|)^k\mathcal{M}^{1/2}$ and take the supreme in v, we obtain that $\|I\|_{H_{l,k}} \le \|h\|_{H_{l,k}}$.

For the second term, we use the property that $\mathcal{K}$ improves integrability, namely

$$\mathcal{K}:\ H_{l,k} \to H_{l,k+1}.$$

Then,

$$\|II\|_{H_{l,k}(t)} \le \int_0^t e^{-(t-s)\nu(v)}\|\mathcal{K}\psi[h]\|_{H_{l.k}}(s)ds \le \|\mathcal{K}\|_{H_{l,k}\to H_{l,k+1}}\int_0^t e^{-(t-s)\nu(v)}\|\psi[h]\|_{H_{l.k-1}}(s)ds.$$

The gain on integrability can be used to control the unbounded ν, as it is easy to see

$$\|\psi[h]\|_{H_{l.k-1}} \le C\|h\|_{H_{l.k}},$$

under the assumption that $\gamma \le 1$. Therefore, we get $\|II\|_{H_{l,k}} \le \|h\|_{H_{l,k}}$.

We conclude that strong solution exists in all time $g \in \mathcal{C}^0(\mathbb{R}^+; H_{l,k}) \cap \mathcal{C}^1(\mathbb{R}^+; H_{l-1.k-1})$ if g^{in} is close enough to $\mathcal{M}$. Through a more careful estimate, one can obtain that g decays to zero exponentially in $H_{l,k}$.

3.2 DiPerna-Lions theory

Ukai's theory works for initial data that is close to equilibrium. The existence theory for large L^1 initial data is due to DiPerna and Lions.

To make sense of the equation for rough initial data, one can consider weak solution, namely f solves the Boltzmann equation

$$\partial_t f + v \cdot \nabla_x f = \mathcal{B}[f, f]$$

in the sense of distribution.

3.2.1 Renormalized solution

Without the smallness assumption, we rely on apriori bounds.

- Mass and energy conservation

$$\int_{\mathbb{T}^D \times \mathbb{R}^D} (1 + |v|^2) f(t, x, v) dx dv = \int_{\mathbb{T}^D \times \mathbb{R}^D} (1 + |v|^2) f^{in}(x, v) dx dv =: E.$$

- H-theorem (relative entropy)

$$\mathcal{H}[f|\bar{\rho}\mathcal{M}](t) \le \mathcal{H}[f^{in}|\bar{\rho}\mathcal{M}] =: H,$$

  where $r\bar{h}o = \int_{\mathbb{T}^D \times \mathbb{R}^D} f(t, x, v) dx dv$ which is conserved in time. Also, we have the bound

$$\int_0^\infty \int_{\mathbb{T}^D} P[f](t, x) dx ds \le \mathcal{H}[f^{in}|\bar{\rho}\mathcal{M}],$$

  where the entropy production P is defined as

$$P[f](t, x) = \frac{1}{4} \int_{\mathbb{S}^{D-1} \times \mathbb{R}^{2D}} \log\left(\frac{f' f'_*}{f f_*}\right) (f' f'_* - f f_*) b(|v - v_*|, \omega) d\omega dv dv_*.$$

Let us consider the mapping $f \mapsto \mathcal{B}[f, f]$. The map is nonlocal in v and has a convolution structure, and it is local in (t, x) as a pointwise product.

The apriori bounds provide L^∞ control in t, and L^1 control in (x, v). While one can make sense of the collision operator as a distribution in t and v, the product of two L^1 function in x is not a distribution. Therefore, we can not make sense of $\mathcal{B}[f, f]$ as a distribution given the aprori bounds.

DiPerna and Lion proposed to write the equation in a different way so that the collision kernel becomes a distribution. Let us formally multiply the Boltzmann equation by $\frac{1}{2\sqrt{1+f}}$. Then we obtain

$$(\partial_t + v \cdot \nabla_x)\sqrt{1 + f} = \frac{1}{2\sqrt{1 + f}} \mathcal{B}[f, f].$$

Lemma 3.4. *Suppose the collision cross section b satisfies*

$$0 \leq \int_{\mathbb{S}^{D-1}} b(z,\omega) d\omega \leq c_b(1+|z|^2).$$

Then, for any $T, R > 0$,

$$\int_0^T dt \int_{\mathbb{T}^D} dx \int_{|v|\leq R} \frac{|\mathcal{B}[f,f](t,x,v)|}{\sqrt{1+f(t,x,v)}} dv \leq H + 2\sqrt{\omega_D c_b EHT(1+R^2)R^D},$$

where ω_D is the area of the unit ball in D dimension.

Proof. First, we recall an elementary inequality

$$(\sqrt{a}-\sqrt{b})^2 \leq \frac{1}{4}(a-b)(\log a - \log b), \quad \forall\, a,b > 0.$$

Therefore, take $a = f'f'_*$ and $b = ff_*$, we obtain

$$\int_{\mathbb{S}^{D-1}\times\mathbb{R}^{2D}} (\sqrt{f'f'_*} - \sqrt{ff_*})^2 b\, d\omega dv dv_* \leq P[f].$$

Note that we can write

$$f'f'_* - ff_* = (\sqrt{f'f'_*} - \sqrt{ff_*})^2 + 2\sqrt{ff_*}(\sqrt{f'f'_*} - \sqrt{ff_*}).$$

Then, using Cauchy-Schwarz inequality, we get

$$\begin{aligned}
&\int_0^T dt \int_{\mathbb{T}^D} dx \int_{|v|\leq R} \frac{|\mathcal{B}[f,f](t,x,v)|}{\sqrt{1+f(t,x,v)}} dv \\
&\leq \int_0^T dt \int_{\mathbb{T}^D} dx \int_{\mathbb{S}^{D-1}\times\mathbb{R}^{2D}} (\sqrt{f'f'_*} - \sqrt{ff_*})^2 b\, d\omega dv dv_* \\
&\quad + \left(\int_0^T dt \int_{\mathbb{T}^D} dx \int_{\mathbb{S}^{D-1}\times\mathbb{R}^{2D}} (\sqrt{f'f'_*} - \sqrt{ff_*})^2 b\, d\omega dv dv_*\right)^{1/2} \times \\
&\quad \left(\int_0^T dt \int_{\mathbb{T}^D} dx \int_{|v|\leq R} dv \int_{\mathbb{S}^{D-1}\times\mathbb{R}^D} \underbrace{\frac{2f}{1+f}}_{\leq 2} f_*\, b\, d\omega dv_*\right)^{1/2} \\
&\leq H + \sqrt{H}\left(2\int_0^T dt \int_{\mathbb{T}^D} dx \int_{|v|\leq R} dv \int_{\mathbb{R}^D} f(v_*) c_b (1 + 2|v|^2 + 2|v_*|^2) dv_*\right)^{1/2} \\
&\leq H + 2\sqrt{\omega_D c_b EHT(1+R^2)R^D}.
\end{aligned}$$

□

The lemma shows that if the initial data has finite energy and entropy, then

$$\frac{\mathcal{B}[f,f]}{\sqrt{1+f}} \in L^1_{loc}(\mathbb{R}_+ \times \mathbb{T}^D \times \mathbb{R}^D). \tag{3.5}$$

The weight $(1+f)^{-1/2}$ compensate the growth of $\mathcal{B}$ when f is large, so that we can make sense of the collision kernel as a distribution.

Therefore, we should study the following notation of *renormalized solution*.

Definition 3.1. A nonnegative function $f \in \mathcal{C}(\mathbb{R}_+; L^1(\mathbb{T}^D \times \mathbb{R}^D)$ is a *renormalized solution* of the Boltzmann equation if (3.5) is satisfied, and for any $\Gamma \in \mathcal{C}^1(\mathbb{R}_+)$ such that

$$\Gamma'(z) \leq \frac{C}{\sqrt{1+z}} \quad \forall\, z \geq 0,$$

one has

$$(\partial_t + v \cdot \nabla_x)\Gamma(f) = \Gamma'(f)\mathcal{B}[f,f],$$

in the sense of distributions on $\mathbb{R}_+ \times \mathbb{T}^D \times \mathbb{R}^D$.

The renormalized solution is weaker than the weak solution. Due to the nonlinear transformation, we give up the usual invariance and conservations.

3.2.2 L^2 velocity averaging

To prove existence of renormalized solution, we will construct a sequence of approximate solutions and show that it converges to a renormalized solution. The main difficulty is the passage to the limit of the collision term. One needs compactness results that can be adapted to the collision integral.

The main idea of gaining compactness is called *velocity averaging*. Our discussion starts with L^2 framework, as Fourier analysis can be used as a clear illustration.

Let us discuss the following question: if $f \in L^2(\mathbb{T}^D)$ and $\mathcal{L}f \in L^2(\mathbb{T}^D)$, then can we obtain compactness or even regularity on f?

If $\mathcal{L} = \Delta$, we know that $f \in H^2(\mathbb{T}^D)$ as the elliptic operator Δ regularizes the solution. However, for $\mathcal{L} = v \cdot \nabla_x$, the hyperbolic operator does not have the regularizing effect.

Exercise 15. *Take $D = 2$, $f(x,v) = H(x_1 v_2 - v_1 x_2)$, where $H(z) = \mathbb{1}_{z>0}$. Prove that $f \in L^2(\mathbb{T}^D \times \mathbb{R}^D)$, $v \cdot \nabla_x f \equiv 0$, but $f \notin L^2(\mathbb{R}^D, H^\epsilon(\mathbb{T}^D))$, for any $\epsilon > 0$.*

The velocity averaging argument says, even if f does not necessarily regularized by the hyperbolic operator, the macroscopic observables $\int_{\mathbb{R}^D} f(t,x,v)\varphi(v)dv$ can be regularized. Note that by the theory of relative entropy, we know that the distance between f and the manifold of local Maxwellians is controlled by entropy production. And local Maxwellians are parametrized by moments on v of f. Therefore, the regularization in macroscopic observables are enough to gain compactness (or regularity).

Let us first state the a version of the Rellich's compactness theorem.

Theorem 3.5. *Let $\{f_\epsilon\}$ be a family of bounded functions in $L^2(\mathbb{T}^D)$. Then, the $\{f_\epsilon\}$ is relative compact in $L^2(\mathbb{T}^D)$ if*

$$\lim_{R\to\infty}\left[\sup_\epsilon \sum_{|k|>R} |\hat{f}_\epsilon(k)|^2\right] = 0.$$

A similar version of the compactness theorem works on $L^2(\mathbb{R}^D)$, where $\hat{f}$ is the continuous Fourier transformation. In this case, one can obtain that $\{f_\epsilon\}$ is relative compact in $L^2_{loc}(\mathbb{R}^D)$.

The L^2 velocity averaging lemma states as follows.

Theorem 3.6 (L^2 velocity averaging)**.** *If $\{f_\epsilon\}$ is a bounded family of functions in $L^2(\mathbb{T}^D\times\mathbb{R}^D)$. Suppose $\{v\cdot\nabla_x f_\epsilon\}$ is also bounded in $L^2(\mathbb{T}^D\times\mathbb{R}^D)$. Then, for all $\varphi\in L^2(\mathbb{R}^D)$, $\{\int_{\mathbb{R}^D}\varphi(v)f_\epsilon(x,v)dv\}$ is relatively compact in $L^2(\mathbb{T}^D)$.*

Proof. Let $h_\epsilon = f_\epsilon + v\cdot\nabla_x f_\epsilon \in L^2(\mathbb{T}^D\times\mathbb{R}^D)$. Take Fourier series in x, we get

$$\hat{h}_\epsilon(k,v) = (1+iv\cdot k)\hat{f}_\epsilon(k,v).$$

Let $\rho_\epsilon(x) = \int_{\mathbb{R}^D}\varphi(v)f_\epsilon(x,v)dv$. Then,

$$\hat{\rho}_\epsilon(k) = \int_{\mathbb{R}^D}\frac{\varphi(v)}{1+iv\cdot k}\hat{h}_\epsilon(k,v)dv.$$

Apply Cauchy-Schwarz inequality, we obatin

$$|\hat{\rho}_\epsilon(k)|^2 \le \underbrace{\left(\int_{\mathbb{R}^D}\frac{|\phi(v)|^2}{1+(v\cdot k)^2}dv\right)}_{:=\Lambda(|k|,\frac{k}{|k|})}\left(\int_{\mathbb{R}^D}|h_\epsilon(k,v)|dv\right),$$

where

$$\Lambda(r,\omega) = \int_{\mathbb{R}^D}\frac{|\varphi(v)|^2}{1+r^2(v\cdot\omega)^2}dv.$$

For any $v\in\mathbb{R}^D$, it is easy to check that $\frac{|\varphi(v)|^2}{1+r^2(v\cdot\omega)^2}\to 0$ as $r\to\infty$, if $v\cdot\omega\neq 0$. Since $\{v\ :\ v\cdot\omega = 0\}$ is a set of measure zero under dv, we get pointwise convergence almost everywhere. Moreover, $\frac{|\varphi(v)|^2}{1+r^2(v\cdot\omega)^2}\le|\varphi(v)|^2\in L^1(\mathbb{R}^D)$. By dominate convergence theorem, we obtain

$$\lim_{r\to\infty}\Lambda(r,\omega) = 0,\quad \forall\,\omega\in\mathbb{S}^{D-1}.$$

As Λ is non-increasing in r and continuous in ω, by Dini's theorem, the convergence is uniform in ω. Finally, we get

$$\sup_\epsilon\sum_{|k|>R}|\hat{\rho}_\epsilon(k)|^2 \le \underbrace{\sup_{\omega\in\mathbb{S}^{D-1}}\Lambda(R,\omega)}_{\to 0,\text{ as }R\to\infty}\cdot\underbrace{\sup_\epsilon\|h_\epsilon\|_{L^2(\mathbb{T}^D\times\mathbb{R}^D)}}_{\text{Bounded}}.$$

Compactness follows from Theorem 3.5. □

There are several remarks to be made about the averaging lemma.

1. One can obtain regularity by additional information on the convergence rate of $\sup_\omega \Lambda(r,\omega)$ as $r \to \infty$. For instance, in Theorem 3.5, it can be shown that $\sup_\omega \Lambda(r,\omega)$ behaves like r^{-1}. Therefore,

$$\|\rho_\epsilon\|^2_{\dot{H}^{1/2}} = \sum_k |k||\hat{\rho}_\epsilon(k)|^2 \le \sum_k \underbrace{|k|\Lambda\left(|k|, \frac{k}{|k|}\right)}_{\text{bounded}} \int_{\mathbb{R}^D} |h_\epsilon(k,v)|^2 dv \le C\|h_\epsilon\|^2_{L^2(\mathbb{T}^D\times\mathbb{R}^D)}.$$

2. The key ingredient of the lemma is that the non-degenerate set is a hyperplane which has Lebesgue measure zero. The lemma won't work with a measure where the hyperplane does not have a zero measure. As an example, discrete Boltzmann equation does not satisfy velocity averaging.

3. The averaging lemma can be extended to L^p, for $p \in (1,\infty)$. One can control the $W^{s,p}$ norm where $s = \min\{1/p, 1/p^*\}$. However, for $p = 1$ and $p = \infty$, the compactness result does not hold.

3.2.3 L^1 velocity averaging

The DiPerna-Lions framework is L^1 based. Unfortunately, the velocity averaging can not be directly applied to L^1 functions.

Here is an example. Let $\{h_n(x,v)\}$ be a sequence of non-negative functions such that $\|h_n\|_{L^1} = 1$, and $h_n \rightharpoonup \delta_{x=0}\otimes\delta_{v=v^*}$ weakly in probability measure. Denote $\{f_n\}$ as the solution of

$$f_n + v\cdot\nabla_x f_n = h_n, \quad \Rightarrow \quad f_n(x,v) = \int_0^\infty e^{-s} h_n(x-sv,v)ds.$$

It is easy to check that

$$\|f_n\|_{L^1} \le \int_0^\infty e^{-s}\|h_n\|_{L^1} ds \le 1, \quad \|v\cdot\nabla_x f_n\|_{L^1} \le \|f_n\|_{L^1} + \|h_n\|_{L^1} \le 2.$$

However, we argue that $\{\int_{\mathbb{R}^D} f_n(x,v)dv\}$ is not relatively compact in L^1. Indeed, let $\varphi = \varphi(x)$ be a smooth test function. Compute

$$\int_{\mathbb{T}^D} \varphi(x)\left(\int_{\mathbb{R}^D} f_n(x,v)dv\right)dx = \int_0^\infty e^{-s}ds \int_{\mathbb{T}^D\times\mathbb{R}^D} \varphi(x)h_n(x,v)dxdv \xrightarrow{n\to\infty} \int_0^\infty e^{-s}\varphi(-sv^*)ds.$$

For $D \ge 2$ and $v^* \ne 0$, the limiting probability measure is singular with respect to Lebesgue measure. Therefore, $\{\int_{\mathbb{R}^D} f_n(x,v)dv\}$ can not be relatively compact in L^1.

The example shows that L^1 velocity averaging breaks down when there is concentration. So, in order to get an L^1 theory, additional assumptions have to be made which exclude the case of concentration.

Let us recall two concepts for a set: *equi-integrability* and *tightness*. Let μ be a Borel measure. We call a set $\mathcal{F} \subset L^1(\mathbb{R}^D; \mu)$ is *equi-integrable* if

$$\lim_{c\to\infty} \sup_{f\in\mathcal{F}} \int_{|f(x)|>c} |f(x)| d\mu(x) = 0.$$

We call $\mathcal{F}$ is *tight* if

$$\lim_{R\to\infty} \sup_{f\in\mathcal{F}} \int_{|x|>R} |f(x)| d\mu(x) = 0.$$

Equi-integrability and tightness prevent concentration and spread-out, respectively. For instance, the set of approximate identity $\eta_n(x) = n^D \eta(nx)$ is not equi-integrable.

With equi-integrability and tightness assumptions, the L^1 velocity averaging argument works as follows.

Theorem 3.7 (L^1 velocity averaging)**.** *If $\{f_\epsilon\}$ is a bounded, equi-integrable and tight family of functions in $L^1(\mathbb{T}^D \times \mathbb{R}^D)$. Suppose $\{v \cdot \nabla_x f_\epsilon\}$ is also bounded in $L^1(\mathbb{T}^D \times \mathbb{R}^D)$. Then, for all $\varphi \in L^\infty(\mathbb{R}^D)$, $\{\int_{\mathbb{R}^D} \varphi(v) f_\epsilon(x,v)dv\}$ is relatively compact in $L^1(\mathbb{T}^D)$.*

Proof. First, we study the resolvant of the transport operator. Let

$$h_\lambda(x,v) = \lambda f(x,v) + v \cdot \nabla_x f(x,v).$$

Then, we have the explicit solution

$$f(x,v) = \int_0^\infty e^{-\lambda t} h_\lambda(x - tv, v) dt.$$

Therefore, $\|f\|_{L^1} \leq \lambda^{-1} \|h_\lambda\|_{L^1}$. Let $\mathcal{R}_\lambda = (\lambda \mathbb{I} + v \cdot \nabla_x)^{-1}$, then

$$\|\mathcal{R}_\lambda\|_{L^1 \to L^1} \leq \frac{1}{\lambda}.$$

Next, we decompose f_ϵ as follows

$$\begin{aligned} f_\epsilon(x,v) =& \mathbb{1}_{|v|>R} f_\epsilon(x,v) + \mathcal{R}_\lambda(\lambda\mathbb{I} + v\cdot\nabla_x)\left(\mathbb{1}_{|v|\leq R} f_\epsilon(x,v)\right) \\ =& \underbrace{\mathbb{1}_{|v|>R} f_\epsilon(x,v)}_{I_\epsilon} + \underbrace{\mathcal{R}_\lambda(v\cdot\nabla_x)\left(\mathbb{1}_{|v|\leq R} f_\epsilon(x,v)\right)}_{II_\epsilon} + \underbrace{\lambda\mathcal{R}_\lambda\left(\mathbb{1}_{|v|\leq R} f_\epsilon(x,v)\right)}_{III_\epsilon}. \end{aligned}$$

Fix any $\eta > 0$. By tightness, we can pick R big enough so that $\|I_\epsilon\|_{L^1} \leq \eta$ for all $f_\epsilon \in \mathcal{F}$. For the second term, by uniform boundedness of $\{v \cdot \nabla_x f_\epsilon\}$, we get

$$\|II_\epsilon\|_{L^1} \leq \frac{1}{\lambda} \|(v\cdot\nabla_x) f_\epsilon\|_{L^1} \leq \frac{C}{\lambda}.$$

So, we can pick λ big enough so that $\|II_\epsilon\|_{L^1} \leq \eta$ for all $f_\epsilon \in \mathcal{F}$.

For the third term, we perform a further decomposition.

$$\lambda\mathcal{R}_\lambda\left(\mathbb{1}_{|v|\le R} f_\epsilon(x,v)\right) = \underbrace{\lambda\mathcal{R}_\lambda\left(\mathbb{1}_{|v|\le R}\mathbb{1}_{|f(x,v)|>c} f_\epsilon(x,v)\right)}_{IV_\epsilon} + \underbrace{\lambda\mathcal{R}_\lambda\left(\mathbb{1}_{|v|\le R}\mathbb{1}_{|f(x,v)|\le c} f_\epsilon(x,v)\right)}_{V_\epsilon}.$$

For the fourth term,

$$\|IV_\epsilon\|_{L^1} \le \lambda \cdot \frac{1}{\lambda}\|\mathbb{1}_{|f_\epsilon(x,v)|>c} f_\epsilon(x,v)\|_{L^1}.$$

By equi-integrability, we can pick c big enough so that $\|IV\|_{L^1} < \eta$.

It is left to control the fifth term V, where the functions $\{\mathbb{1}_{|v|\le R}\mathbb{1}_{|f_\epsilon(x,v)|\le c} f_\epsilon(x,v)\}$ are uniformly bounded and has a uniform compact support. We can apply L^2 velocity averaging lemma to this part. Indeed, one can check

$$\|V_\epsilon\|_{L^2} \le \lambda\|\mathcal{R}_\lambda\|_{L^2\to L^2}\|\mathbb{1}_{|v|\le R}\mathbb{1}_{|f_\epsilon(x,v)|\le c} f_\epsilon(x,v)\|_{L^2} \le \left(c\|f_\epsilon\|_{L^1}\right)^{1/2}.$$

Also, note that $v\cdot\nabla_x\mathcal{R}_\lambda = \mathbb{I} - \lambda\mathcal{R}_\lambda$, we get

$$\|(v\cdot\nabla_x)V_\epsilon\|_{L^2} \le \lambda\left(1+\lambda\|\mathcal{R}_\lambda\|_{L^2\to L^2}\right)\|\mathbb{1}_{|v|\le R}\mathbb{1}_{|f_\epsilon(x,v)|\le c} f_\epsilon(x,v)\|_{L^2} \le 2\lambda\left(c\|f_\epsilon\|_{L^1}\right)^{1/2}.$$

Applying Theorem 3.6 we get $\mathcal{V} = \{\int_{\mathbb{R}^D}\varphi(v)V_\epsilon dv\}$ is relatively compact in $L^2(\mathbb{T}^D)$. Since V_ϵ is uniformly bounded, $\mathcal{V}$ is relatively compact in $L^1(\mathbb{T}^D)$ as well.

To sum up, we have

$$\left\{\int_{\mathbb{R}^D}\varphi(v)f_\epsilon(x,v)dv\right\} \subset B(3\eta\|\varphi\|_{L^\infty})_{L^1(\mathbb{T}^D)} + \mathcal{V}$$

which is relatively compact in $L^1(\mathbb{T}^D)$. □

Some remarks are stated as follows.

1. If the test function φ is compactly supported, then $I_\epsilon = 0$ if one pick R large enough. Therefore, tightness assumption is no longer needed.

2. $\{f_\epsilon\}$ is equi-integrable in v implies $\{f_\epsilon\}$ is equi-integrable in (x,v), due to spatial mixing property of free transport operator. Therefore, we can relax the equi-integrability assumption to v variable only.

3. A similar argument can be made for time-dependent problem. We omit the details here.

3.2.4 Existence of renormalized solutions

To construct a solution of Boltzmann equation, we start with a sequence of approximate solutions. One choice of the approximating sequence $\{f_n\}$ solves

$$\partial_t f_n + v\cdot\nabla_x f_n = \frac{\mathcal{B}[f_n,f_n]}{1+\frac{1}{n}\int_{\mathbb{R}^D} f_n dv}. \tag{3.6}$$

We first state the existence theorem for the approximate solutions. For simplicity, we assume the collision cross section $b \in L^\infty(\mathbb{R}^D \times \mathbb{S}^{D-1})$. For the general case, one can perform a truncation and pass to the limit. Denote

$$\bar{b}(z) = \int_{\mathbb{S}^{D-1}} b(z,\omega)d\omega, \quad \text{and} \quad \mathcal{A}[f](t,x,v) = f *_v \bar{b} = \int_{\mathbb{R}^D} f(t,x,v_*)\bar{b}(v-v_*)dv_*.$$

Proposition 3.8. *Consider the Cauchy problem of* (3.6) *with initial data* $f^{in}(x,v)$*. There exists a unique solution for the initial value problem.*

Sketch of the proof. Let us rewrite (3.6) as follows.

$$\begin{aligned}\partial_t f_n + & v\cdot\nabla_x f_n + n\|b\|_{L^\infty} f_n \\ = & \frac{\int_{\mathbb{R}^D} f_n' f_{n*}' \bar{b}(v-v_*)dv_*}{1+\frac{1}{n}\int_{R^D} f_n dv} + \frac{\int_{\mathbb{R}^D} f_{n*}(\|b\|_{L^\infty} - \bar{b}(v-v_*))dv_*}{1+\frac{1}{n}\int_{R^D} f_n dv} f_n + \frac{n\|b\|_{L^\infty}}{1+\frac{1}{n}\int_{R^D} f_n dv} f_n.\end{aligned}$$

The left hand side of the equation generates a $\mathcal{C}^0$-semigroup in $L^1(\mathbb{T}^D \times \mathbb{R}^D)$. The right hand side is Lipschitz with respect to f_n, in $L^1_+(\mathbb{T}^D \times \mathbb{R}^D)$. One can apply fix point argument to prove existence and uniqueness of solutions. □

Exercise 16. *Complete the proof of Proposition 3.8.*

Let us denote the collision operator in (3.6) as $\mathcal{C}_n[f_n]$. Unlike the renormalized collision integral $\Gamma'(f)\mathcal{B}[f,f]$, $\mathcal{C}_n$ has the same collision invariant as $\mathcal{B}$. Hence all the conservations hold.

$$\int_{\mathbb{T}^D\times\mathbb{R}^D} (1+|v|^2) f_n(t,x,v)dxdv = \int_{\mathbb{T}^D\times\mathbb{R}^D} (1+|v|^2) f^{in}(x,v)dxdv = E.$$

The H-theorem is also available in the following form.

$$\mathcal{H}[f_n|\bar{\rho}\mathcal{M}](t) + \int_0^t \int_{\mathbb{T}^D} \frac{P[f_n](t,x)}{1+\frac{1}{n}\int_{R^D} f_n dv} dxds \le \mathcal{H}[f^{in}|\bar{\rho}\mathcal{M}] = H.$$

Exercise 17. *If* $\{f_n\}$ *satisfies the energy and entropy bounds, then* $\{f_n\}$ *is bounded in* $L^1_{loc}(\mathbb{R}^+ \times \mathbb{T}^D \times \mathbb{R}^D)$*, equi-integrable and tight.*

By Dunford-Pettis theorem, $\{f_n\}$ is relatively compact in $L^1_{loc}(\mathbb{R}^+ \times \mathbb{T}^D \times \mathbb{R}^D)$ under weak topology. Therefore, modulo subtracting a subsequence, we have

$$f_n \rightharpoonup f, \quad \text{weakly in } L^1_{loc}(\mathbb{R}^+ \times \mathbb{T}^D \times \mathbb{R}^D).$$

We now aim to prove that f is a renormalized solution of Boltzmann equation.

Choose a family of nonlinear renormalization functions

$$\Gamma_\delta(z) = \frac{1}{\delta}\log(1+\delta z).$$

The renormalized approximated Boltzmann equation reads

$$(\partial_t + v\cdot\nabla_x)\Gamma_\delta(f_n) = \frac{\mathcal{C}_n[f_n]}{1+\delta f_n}.$$

Since $\Gamma_\delta(z) \le z$ for all $z \ge 0$ and $\delta > 0$, it is easy to see that $\{\Gamma_\delta(f_n)\}$ is also bounded, equi-integrable and tight.

Decompose $\mathcal{C}_n$ into two parts $\mathcal{C}_n = \mathcal{C}_n^+ - \mathcal{C}_n^-$, where

$$\mathcal{C}_n^+[f] = \frac{\int_{\mathbb{R}^D} f'f'_*\bar{b}(v-v_*)dv_*}{1+\frac{1}{n}\int_{\mathbb{R}^D} f_* dv_*}, \quad \mathcal{C}_n^-[f] = \frac{\int_{\mathbb{R}^D} ff_*\bar{b}(v-v_*)dv_*}{1+\frac{1}{n}\int_{\mathbb{R}^D} f_* dv_*} = \frac{f\mathcal{A}[f]}{1+\frac{1}{n}\int_{\mathbb{R}^D} f_* dv_*}.$$

Lemma 3.9. *For all* $\delta, T, R > 0$, $\dfrac{\mathcal{C}_n^{\pm}[f_n]}{1+\delta f_n}$ *is bounded in* $L^1((0,T)\times\mathbb{T}^D\times B(R)_{\mathbb{R}^D})$, *where* $B(R)_{\mathbb{R}^D} = \{z\in\mathbb{R}^D \ : \ |z|\le R\}$.

Proof. We start with $\mathcal{C}_n^-$.

$$\frac{C_n^-[f_n]}{1+\delta f_n} \le \frac{f_n\mathcal{A}[f_n]}{1+\delta f_n} \le \frac{1}{\delta}\mathcal{A}[f_n] \le \frac{1}{\delta}\|b\|_{L^\infty}\int_{\mathbb{R}^D} f_n dv.$$

Therefore, we get

$$\int_0^T dt\int_{\mathbb{T}^D} dx\int_{|v|\le R}\frac{C_n^-[f_n]}{1+\delta f_n}dv \le T\omega_D R^D\delta^{-1}\|b\|_{L^\infty}\int_{\mathbb{T}^D\times\mathbb{R}^D} f_n dxdv \le T\omega_D R^D\delta^{-1}\|b\|_{L^\infty}E.$$

For $\mathcal{C}_n^+$, since $\mathcal{C}_n^+ \le \mathcal{C}_n^- + |\mathcal{C}_n|$, we apply the bound on $\mathcal{C}_n^-$ and Lemma 3.4 to get a bound for $\mathcal{C}_n^+$. □

Now, we are in position to apply L^1 velocity averaging (Theorem 3.7) and obtain strong compactness of $\{\int\varphi(v)\Gamma_\delta[f_n]dv\}$.

Lemma 3.10. *For any* $\varphi\in L^\infty(\mathbb{R}^D)$,

$$\int_{\mathbb{R}^D}\varphi(v)f_n dv - \int_{\mathbb{R}^D}\varphi(v)\Gamma_\delta[f_n]dv \to 0$$

in $L^1((0,T)\times\mathbb{T}^D)$ *as* $\delta\to 0$ *uniformly in* n.

Proof. Using the elementary inequality $0 \le z - \log(1+z) \le \frac{z^2}{2}$, one can obtain

$$0 \le f_n - \Gamma_\delta[f_n] \le \frac{\delta f_n^2}{2}.$$

Decompose $\mathbb{R}^D$ into two parts $\{|f_n(x)| > c\}$ and $\{|f_n(x)| \le c\}$. For the first part,

$$\left|\int_{|f_n(x)|>c}\varphi(v)(f_n - \Gamma_\delta[f_n])dv\right| \le \left|\int_{|f_n(x)|>c}\varphi(v)f_n dv\right| \le \|\varphi\|_{L^\infty(\mathbb{R}^D)}\int_{|f_n(x)|>c} f_n dv.$$

Given any $\epsilon > 0$, we can find a large enough c such that the $L^1((0,T)\times\mathbb{T}^D)$ norm of this term is bounded by $\epsilon/2$ for all n, due to equi-integrability. For the second part,

$$\left|\int_{|f_n(x)|\le c}\varphi(v)(f_n-\Gamma_\delta[f_n])dv\right| \le \left|\int_{|f_n(x)|\le c}\varphi(v)\frac{\delta f_n^2}{2}dv\right| \le \|\varphi\|_{L^\infty(\mathbb{R}^D)}\frac{\delta c}{2}\int_{\mathbb{R}^D} f_n dv.$$

Pick δ small enough, the $L^1((0,T)\times\mathbb{T}^D)$ norm of this term is bounded by $\epsilon/2$ for all n, using conservation of mass. □

A direct consequence of Lemma 3.10 is, the set $\{\int\varphi(v)f_n dv\}$ is relatively compact in $L^1((0,T)\times\mathbb{T}^D)$, for any $\varphi\in L^\infty(\mathbb{R}^D)$. The strong compactness allows us to pass to the limit of the collision integral.

To see this, we first state a version of the product limit theorem.

Proposition 3.11. *Let X, Y be measure spaces with finite positive measure. Let a_n and b_n be sequences of measurable functions in $X\times Y$. Assume $a_n \rightharpoonup a$ in weakly in $L^1(X\times Y)$, b_n is uniformly bounded in $L^\infty(X\times Y)$ and $b_n\to b$ almost everywhere. Then,*

$$a_n b_n \rightharpoonup ab \quad \textit{weakly in } L^1(X).$$

Moreover, if for any $\varphi\in L^\infty(Y)$, $\int_Y a_n(x,y)\varphi(y)dy \to \int_Y a(x,y)dy$ strongly in $L^1(X)$. Then,

$$\int_Y a_n(x,y)b_n(x,y)\varphi(y)dy \to \int_Y a(x,y)b(x,y)\varphi(y)dy \quad \textit{strongly in } L^1(X).$$

Exercise 18. *Prove Proposition 3.11.*

We are ready to prove the following limit for the collision integral.

Theorem 3.12. *For any $\varphi\in L^\infty(\mathbb{R}^D)$,*

$$\frac{\int_{\mathbb{R}^D}\varphi(v)\mathcal{C}_n^\pm[f_n]dv}{1+\int_{\mathbb{R}^D} f_n dv} \to \frac{\int_{\mathbb{R}^D}\varphi(v)\mathcal{B}^\pm[f,f]dv}{1+\int_{\mathbb{R}^D} f dv}, \quad \textit{in } L^1((0,T)\times\mathbb{T}^D). \tag{3.7}$$

Proof. We start with the loss term $\mathcal{C}_n^-$. Recall

$$\mathcal{C}_n^-[f_n] = \frac{\mathcal{A}[f_n]}{1+\frac{1}{n}\int_{\mathbb{R}^D} f_n dv} f_n.$$

Apply velocity averaging, we get

$$\frac{\mathcal{A}[f_n]}{1+\frac{1}{n}\int_{\mathbb{R}^D} f_n dv} = \frac{\bar{b} *_v f_n}{1+\frac{1}{n}\int_{\mathbb{R}^D} f_n dv} \to \bar{b} *_v f, \quad \text{almost everywhere in } (t,x,v),$$

modulo subtracting a subsequence. Likewise, we have

$$\frac{\mathcal{A}[f_n]}{(1+\frac{1}{n}\int_{\mathbb{R}^D} f_n dv)(1+\int_{\mathbb{R}^D} f_n dv)} \to \frac{\bar{b} *_v f}{1+\int_{\mathbb{R}^D} f_n dv}, \quad \text{almost everywhere in } (t,x,v).$$

Moreoever, we have the L^∞ bound on v

$$\left\| \frac{\mathcal{A}[f_n]}{(1+\frac{1}{n}\int_{\mathbb{R}^D} f_n dv)(1+\int_{\mathbb{R}^D} f_n dv)} \right\|_{L^\infty(\mathbb{R}^D)} \le \frac{\|b\|_{L^\infty}\int_{\mathbb{R}^D} f_n dv}{1+\int_{\mathbb{R}^D} f_n dv} \le \|b\|_{L^\infty}.$$

Hence,

$$\left\| \frac{\mathcal{A}[f_n]}{(1+\frac{1}{n}\int_{\mathbb{R}^D} f_n dv)(1+\int_{\mathbb{R}^D} f_n dv)} \right\|_{L^\infty(\mathbb{R}^+\times\mathbb{T}^D\times\mathbb{R}^D)} \le \|b\|_{L^\infty}.$$

Apply proposition 3.11, we get

$$\frac{\mathcal{C}_n^-[f_n]}{1+\int_{\mathbb{R}^D} f_n dv} \rightharpoonup \frac{\bar{b} *_v f}{1+\int_{\mathbb{R}^D} f_n dv} f = \frac{\mathcal{B}^-[f,f]}{1+\int_{\mathbb{R}^D} f_n dv}, \quad \text{in } L^1((0,T)\times\mathbb{T}^D\times B(R)_{\mathbb{R}^D}).$$

For φ with a compact support, we take R large enough so that $supp(\varphi) \subset B(R)$. (3.7) can be derived by integrating in v againest the test function φ. For general $\varphi \in L^\infty(\mathbb{R}^D)$, the same limit holds due to tightness of f_n.

Next, we consider the gain term $\mathcal{C}_n^+$.

$$\mathcal{C}_n^+[f_n] = \frac{\int_{\mathbb{S}^{D-1}\times\mathbb{R}^D} f_n' f_{n*}' b \, d\omega dv_*}{1+\frac{1}{n}\int_{\mathbb{R}^D} f_n dv}.$$

Perform a change of variable $(v, v_*) \to (v', v_*')$, we get

$$\int \varphi(v)\mathcal{C}_n^+[f_n]dv = \frac{1}{1+\frac{1}{n}\int_{\mathbb{R}^D} f_n dv}\int_{\mathbb{S}^{D-1}\times\mathbb{R}^{2D}} \varphi(v') f_n f_{n*} b \, d\omega dv dv_* = \frac{\int_{\mathbb{R}^D} \tilde{\mathcal{A}}[f_n] f_n dv}{1+\frac{1}{n}\int_{\mathbb{R}^D} f_n dv}.$$

where

$$\tilde{\mathcal{A}}[f] := \int_{\mathbb{S}^{D-1}\times\mathbb{R}^D} \varphi(v') f_* b \, d\omega dv_*.$$

Then, we proceed similarly as the loss term:

$$\frac{\tilde{\mathcal{A}}[f_n]}{(1+\frac{1}{n}\int_{\mathbb{R}^D} f_n dv)(1+\int_{\mathbb{R}^D} f_n dv)} \to \frac{\tilde{\mathcal{A}}[f]}{1+\int_{\mathbb{R}^D} f_n dv}, \quad \text{almost everywhere in } (t,x,v).$$

$$\left\| \frac{\tilde{\mathcal{A}}[f_n]}{(1+\frac{1}{n}\int_{\mathbb{R}^D} f_n dv)(1+\int_{\mathbb{R}^D} f_n dv)} \right\|_{L^\infty(\mathbb{R}^+\times\mathbb{T}^D\times\mathbb{R}^D)} \le \|\varphi\|_{L^\infty}\|b\|_{L^\infty}.$$

Apply proposition 3.11 and pick the test function to be 1. We end up with (3.7). □