AMSC/CMSC460 Computational Methods **Spring 2015**

Homework 6-7, Due on Tuesday, April 7, 2015

Problem 3 and 5 are *optional*. You only have to submit Problem 1,2,4 and 6.

1. *(Linear splines)* Let $f(x) = x^3$ in the interval $x \in [0, 1]$. $\{x_k\}_{k=0}^m$ are $m+1$ equally distributed nodes (or knots):

$$x_k = \frac{k}{m}, \quad k = 0, \cdots, m.$$

a). $s_L(x)$ is the linear interpolating spline for f. We write $s_L(x) = \sum_{k=0}^{m} a_k \varphi_k(x)$, where φ_k are basis functions (in this case, hat functions). What are the coefficient a_k? Find $a_0, \cdots, a_5$ for the case $m = 5$.

b). $s(x)$ is the linear spline which minimize $\|f - s\|_2$. We write $s(x) = \sum_{k=0}^{m} \alpha_k \varphi_k(x)$. How to get the coefficient α_k? To answer this question, please follow Exercise 11.4 in Suli's book. Calculate A_{ij} by hand explicitly, and compare with your result with Matlab script `linearspline.m`. Solve the linear system $A\alpha = \mathbf{b}$ for $m = 5$, using your code on Thomas algorithm, and find $\alpha_0, \cdots, \alpha_5$.

2. *(Least square polynomial approximation)* Let $f(x) = x^4$. Find the cubic polynomial $p_3(x)$ which minimizes $\|f - p_3\|_{L^2([-1,1])}$.
Hint: You might make use of Legendre polynomials $\{P_n(x)\}_{n=0}^\infty$, which are orthogonal with respect to L^2 norm.

$$\int_{-1}^{1} P_m(x) P_n(x) dx = \begin{cases} 0 & m \neq n \\ \frac{2}{2m+1} & m = n \end{cases}.$$

The first 4 Legendre polynomials is given as below:

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{2}(3x^2 - 1), \quad P_3(x) = \frac{1}{2}(5x^3 - 3x).$$

3. $^{(\star)}$ *(Natural cubic spline)* The interpolating cubic spline is a $\mathcal{C}^2$ function which is a piecewise cubic polynomial, and it interpolates given function f at given set of knots $\{x_i\}_{i=0}^m$. Since there are 2 extra degrees of freedom, different boundary conditions can be imposed.

a). Write $s(x)$ in the form

$$s(x) = \frac{(x_i - x)^3}{6h_i}\sigma_{i-1} + \frac{(x - x_{i-1})^3}{6h_i}\sigma_i + \alpha_i(x - x_{i-1}) + \beta_i(x_i - x),$$

for $x \in [x_{i-1}, x_i]$, and $h_i := x_i - x_{i-1}$. Check that $s''(x_i-) = s''(x_i+) = \sigma_i$, and express α_i, β_i in terms of $\{\sigma_i\}_i$ and $\{f_i\}_i$.

b). Express $s'(x)$ in terms of $\{\sigma_i\}_i$. What is the condition $s'(x_i-) = s'(x_i+)$ in $\{\sigma_i\}_i$? You should obtain a linear system of $\{\sigma_i\}_i$ with $m + 1$ unknowns and $m - 1$ equations.

c). One natural way to impose boundary condition is $s''(x_0) = s''(x_m) = 0$. Write the linear system $A\sigma = b$ that consists $m-1$ interior equations and the two boundary conditions. Here, $\sigma = (\sigma_0, \cdots, \sigma_m)^T$ is a column vector with $m+1$ entries. Please specify matrix A and vector b. Is the matrix A sparse or not?

d). Consider another set of boundary condition $s'(x_0) = s'(x_m) = 0$. In this case, what does the linear system look like? Write the matrix A and vector b.

e). There is another so-called periodic boundary condition, where $s^{(k)}(x_0) = s^{(k)}(x_m)$ for $k = 0, 1, 2$. In this case, x_0 and x_m are considered as a same point. Now, we denote $\sigma = (\sigma_1, \cdots, \sigma_m)^T$ be a column vector with m entries. (Note here we omit σ_0 as it is the same as σ_m.) Write out the system $A\sigma = b$.

f). Read section 3.5 on numerical implementations for cubic spline in Moler's book.
(The following is *optional*. No need to submit) Write a Matlab code to find interpolating cubic spline, under different setups on boundary conditions.

4. *(Simpson's rule)* Let f be a $\mathcal{C}^4$ function defined in an interval $[a, b]$. $P_2(x)$ is a polynomial of degree 2 that interpolates f at points $x_0 = a, x_1 = (a+b)/2$ and $x_2 = b$. Simpson's rule is a quadrature rule that approximate $\int_a^b f(x)dx$ by $\int_a^b P_2(x)dx$.

a). Check that the approximated value is in the following form

$$\int_a^b f(x)dx \approx \int_a^b P_2(x)dx = \frac{b-a}{6}\left[f(x_0) + 4f(x_1) + f(x_2)\right].$$

b). Φ is the cumulative distribution function for standard normal distribution, given as

$$\Phi(x) = \frac{1}{2} + \frac{1}{\sqrt{2\pi}}\int_0^x e^{-\frac{y^2}{2}}dy.$$

Use Simpson's rule to approximate $\Phi(1)$. What is the approximate value? Use Matlab to check the error (take `normcdf(1)` as the "exact" value of $\Phi(1)$).

c). Read the proof of Theorem 7.2 in Suli's book, and use the result to obtain an error bound for your approximation. Verify that the actual error is smaller than this error bound.

d). Let $\{x_i\}_{i=0}^{2m}$ be equally distributed points in $[a, b]$, namely

$$x_i = a + ih, \quad h = \frac{b-a}{2m}.$$

(Note here we redefine x_0, x_1, x_2.) Write a composite Simpson rule. Use Matlab to approximate $\Phi(1)$ by the rule with $m = 2^s$ for $s = 1, 2, \cdots, 8$. Store the error as a sequence (vector) $\{e(s)\}_{s=1}^8$. Display the sequence $\log_2(e(s))$. What do you observe?

Hint: you should observe the difference between the 2 adjacent terms are very close to -4. Can you explain why this happens?

5. $^{(\star)}$*(Quadrature by Hermite interpolation)* Finish Exercise 7.11 in Suli's book in quadrature obtained by Hermite interpolation. What is the corresponding composite rule?

6. *(Gauss quadrature)* The goal of the exercise is to use Gauss quadrature to solve the integral

$$\mathbb{E}(f(X)) = \int_{-\infty}^{\infty} f(x) \left(\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \right) dx,$$

representing the expectation of $f(X)$ where X is a random variable that has a standard normal distribution.
To this end, we introduce *Hermite polynomials* $\{H_n(x)\}_{n=0}^{\infty}$, which are orthogonal with respect to the weight $w(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$, namely,

$$\int_{-\infty}^{\infty} H_m(x) H_n(x) \left(\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \right) dx = \begin{cases} n! & m = n \\ 0 & m \neq n \end{cases}.$$

The first four Hermite polynomials are

$$H_0(x) = 1, \quad H_1(x) = x, \quad H_2(x) = x^2 - 1, \quad H_3(x) = x^3 - 3x.$$

a). We approximate $\mathbb{E}(f(X))$ by Gauss quadrature with 3 nodes, namely

$$\int_{-\infty}^{\infty} f(x) \left(\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \right) dx \approx w_0 f(x_0) + w_1 f(x_1) + w_2 f(x_2).$$

Find the nodes $\{x_i\}_{i=0}^{2}$ and weights $\{w_i\}_{i=0}^{2}$ such that the approximation is precise for all $f \in \mathbb{P}_5$.

b). Calculate (by hand) the fourth moment of standard normal distribution $\mathbb{E}(X^4)$ by the quadrature you obtained in a). By fourth moment, we mean $f(x) = x^4$.

Hint: Since $f \in \mathbb{P}_5$, we should expect to get the precise value $\mathbb{E}(X^4) = 3$.

c). Use the same quadrature to approximate $\mathbb{E}(|X|^3)$, where $f(x) = |x|^3$. Do you get the exact value $\mathbb{E}(|X|^3) = \sqrt{\frac{8}{\pi}}$? Briefly explain why you get or not get the precise value.