AMSC/CMSC460 Computational Methods **Spring 2015**

Homework 3, Due on Thursday, February 26, 2015

Problem 3 and 4 are *optional*. You only have to submit problem 1 and 2.

1. *(Newton's method)* Write a Matlab code for Newton's iteration method $x_{k+1} = x_k - \dfrac{f(x_k)}{f'(x_k)}$. (You can get a reference code in Moler's book, pp. 119.)

a). Find a root of the zeroth-order Bessel function $J_0(x) = \displaystyle\sum_{m=0}^{\infty} \frac{(-1)^m}{m!\ \Gamma(m+1)} \left(\frac{x}{2}\right)^{2m}$. You can directly use Matlab function `besselj` to represent J_0 and J_0' as below:

```
f = @(x)besselj(0,x);
fprime = @(x) - besselj(1,x);
```

as Bessel function has the property $J_0'(x) = -J_1(x)$. Test with $x_0 = 4$ and $x_0 = 5$.

b). Find a root for $f(x) = x^2 - 187$. Start with $x_0 = 187$. Use the numerical result to verify that Newton's method has quadratic convergence rate:
Compute $\dfrac{e_{k+1}}{e_k^2}$ and check if this number converges to a positive constant. What number do you expect (calculate the number by hand)? Does it supported by the numerical result?

c). Find a root for $f(x) = (x^2 - 187)^2$. Start with $x_0 = 187$. Note the root is still $\sqrt{187}$. Check $\dfrac{e_{k+1}}{e_k^2}$ to see what happened. Expain why Newton's method only provides a linear convergence. Compute $\dfrac{e_{k+1}}{e_k}$ to verify that the convergence is linear. Calculate $\displaystyle\lim_{k\to\infty} \frac{e_{k+1}}{e_k}$, and check it with the numerical result.

2. *(Secant method)*

a). Prove that secant method has a rate of convergence $q = \frac{1}{2}(1 + \sqrt{5})$. You can follow the storyline of excercise 1.10 in Suli's book.

b). Write a Matlab code on secant method and test $f(x) = x^2 - 187$ with $x_0 = 187$, $x_1 = 185$. Check the numerical convergence rate.

3. $^{(\star)}$ *(Square root method)* Consider the following iteration procedure

$$x_{k+1} = x_k - sgn(f'(x_k)) \frac{f(x_k)}{\sqrt{f'(x_k)^2 - f(x_k)f''(x_k)}}.$$

This method is called *square root method*. Prove that this method has a local cubic convergence if $f'(x_*)^2 - f(x_*)f''(x_*) > 0$.
Hint: write the scheme as $x_{k+1} = g(x_k)$, *and verify that* $g(x_*) = x_*,\ g'(x_*) = 0$ *and* $g''(x_*) = 0$.

4. $^{(\star)}$ *Modified Newton's method* Consider the following iterative method.

$$x_{k+1} = x_k - \frac{f(x_k)^2}{f(x_k + f(x_k)) - f(x_k)}, \quad g(x) = x - \frac{f(x)^2}{f(x+f(x)) - f(x)}.$$

Assume x_* is a root of a smooth function $f \in C^\infty$ with $f'(x_*) \neq 0$ and $f''(x_*) \neq 0$. Prove that the method has a local convergence. What is the convergence rate α? Compute the leading coefficient $\lim_{k\to\infty} \frac{e_{k+1}}{e_k^\alpha}$.

Hint: Prepare for heavy computation. Following the steps below might lower your computational burden. (Note: you can try alternative ways, e.g. compute $g(x_*), g'(x_*)$ and $g''(x_*)$ directly, but keep in mind it would cost a lot of time.)

- *Take $F(x) = \frac{f(x+f(x)) - f(x)}{f(x)}$ and express g in terms of f and F.*
- *Taylor expand $f(x + f(x))$ around x and write F as a infinite series. Check that F has the following form:*
$$F(x) = \sum_{n=0}^{\infty} \frac{f^{(n+1)}(x)}{(n+1)!} f(x)^n.$$
- *Compute $F'(x), F''(x)$, and evaluate F, F', F'' at $x = x_*$. Note that $f(x_*) = 0$.*
- *Given $F(x_*), F'(x_*), F''(x_*)$, compute $g(x_*), g'(x_*)$ and $g''(x_*)$. The local convergence is given by checking $g(x_*) = x_*$.*
- *Conclude with the convergence rate α and leading coefficient $\lim_{k\to\infty} \frac{e_{k+1}}{e_k^\alpha}$.*