AMSC/CMSC 460 Computational Methods

Final Exam, Monday, May 18, 2015 Name: ______________________

Show all work clearly and in order, and circle your final answers. Justify your answers algebraically whenever possible. Use no books, calculators, cellphones, communication with others, etc, except two formula sheets (A4 one-sided) prepared by yourself. You have 120 minutes to take this 210 point exam. If you get more than 200 points, your grade will be 200.

1. (40 points) Mark each of the following statements T (True) or F (False).

   [28] This part contains 7 statements. You will get 4 points for each correct answer, -1 points for each wrong answer, and 0 point for leaving it blank.

   (a) ______ Let $\|\cdot\|_p$ be the matrix norm induced by the corresponding vector p-norm. Then the three norms $\|\cdot\|_1, \|\cdot\|_2$ and $\|\cdot\|_\infty$ are equivalent.

   (b) ______ QR factorization of a matrix is unique, namely, for any matrix A, there exists only one pair of orthonormal matrix Q and upper triangular matrix R such that $A = QR$.

   (c) ______ Bisection method has a linear convergence rate.

   (d) ______ Natural cubic spline is smoother than Hermite cubic spline.

   (e) ______ Gauss quadrature uses equally distributed nodes.

   (f) ______ Explicit schemes for initial value problems of first order ODE can never be A-stable.

   (g) ______ The best way to solve a stiff ODE system is to use higher order explicit Runge-Kutta methods.

   [12] For the following 5 statements, choose 3 (and ONLY 3) to answer.

   (h) ______ Iterative methods are more efficient than Gauss elimination when the linear system is large and sparse.

   (i) ______ A similar transform does not change the eigenvalues and the corresponding eigenvectors of the matrix.

   (j) ______ Normalized Chebyshev polynomial of degree n (namely T_n divided by its leading coefficient 2^{n-1}) minimizes the L^∞-norm in $[-1, 1]$, among all the monic polynomials of degree n.

   (k) ______ Richardson extrapolation is used to improve the rate of convergence of a sequence.

   (l) ______ The shooting method can be used to solve second order ODE with Neumann or Robin boundary condition.

2. (20 points) Consider the following Matlab code.

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A = [1 2 2; 1 3 3;2 4 0];
c = cond(A(1:2,1:2),1)
[L,U,P] = lu(A)
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Find the output c, L, U, P by hand.

3. (20 points) (a) [10] Complete the following QR decomposition

$$A = \begin{pmatrix} -1 & -3 & 0 & 0 \\ -1 & 1 & 4 & 4 \\ 1 & 3 & -2 & -2 \\ 1 & -1 & -2 & 2 \end{pmatrix} = \begin{pmatrix} * & -.5 & * & .5 \\ * & .5 & * & .5 \\ * & .5 & * & .5 \\ * & -.5 & * & .5 \end{pmatrix} \begin{pmatrix} -2 & * & * & * \\ * & 4 & * & * \\ * & * & -2 & * \\ * & * & * & 2 \end{pmatrix}.$$

   (b) [10] Use the decomposition in (a) to find a vector $x \in \mathbb{R}^3$ which minimize $\|Ax - b\|_2$ where $b = [1, 0, -1, 0]^T$.

4. (20 points) Suppose $f(x) = (x^2 - 187)^2$, in $(0, \infty)$. The goal is to use iterative scheme to approximate the root of f, which is $x_* = \sqrt{187}$.
   (a) [5] Write the Newton's iteration for this specific f.
   (b) [5] Check the consistency condition: if x_k converges, then the limit must be a root of f.
   (c) [10] What is the convergence rate of the scheme? Verify your statement rigorously.

5. (20 points) Let $f(x) = \dfrac{2}{1+x^2}$.
   (a) [10] Find a polynomial p_3 of degree 3 which is a Hermite interpolation of f (namely it interpolates both f and f') at nodes -1 and 1. Simplify your answer in the form of $\sum_{k=0}^{4} c_k x^k$.
   (b) [10] Find an upper bound of the error $\|f(x) - p_3(x)\|_{L^\infty([-1,1])}$.
   For your convenience, $\max_{x\in[-1,1]} |f^{(n)}(x)| = |f^{(n)}(0)| = 2 \cdot n!$, for all integers $n \geq 0$.

6. (20 points) Let $f(x) = x^5 - 2x^4$. Find the quadratic polynomial $p_2(x)$ which minimizes

$$\int_{-\infty}^{\infty} \big(f(x) - p_2(x)\big)^2 e^{-\frac{x^2}{2}}\, dx.$$

Hint: Hermite polynomials $\{H_n(x)\}_{n=0}^{\infty}$ *are orthogonal with respect to:*

$$\langle H_m, H_n \rangle = \int_{-\infty}^{\infty} H_m(x) H_n(x) e^{-\frac{x^2}{2}}\, dx = \begin{cases} 0 & m \neq n \\ \sqrt{2\pi}\, n! & m = n \end{cases}.$$

They can be defined recursively as $H_{n+1}(x) = xH_n(x) - H_n'(x)$*, with* $H_0(x) = 1$*.*
The following decomposition on f *could save your computational load significantly:*

$$f(x) = H_5(x) - 2H_4(x) + 10H_3(x) - 12H_2(x) + 15H_1(x) - 6H_0(x).$$

7. (15 points) Suppose a quadrature rule $I[f]$ has the following error bound

$$E = \left| \int_a^b f(x)dx - I[f] \right| \leq C(b-a)^7,$$

where C is a constant which depends on f. Consider the following composite rule: let $\{x_i\}_{i=0}^N$ be equally distributed nodes in $[a,b]$, namely $x_i = a + ih$ where $h = (b-a)/N$. Apply quadrature rule $I[f]$ on each interval $[x_{i-1}, x_i]$, for $i = 1, \cdots, N$. Then, sum up the integrands on all intervals.

- *Derive* an error bound E_N for the composite quadrature rule $I_N[f]$.
- What is the rate of convergence for the composite rule?

8. (20 points) We approximate the integrand $\int_{-1}^1 f(x)dx$ by a Gauss quadrature rule $Q[f]$:

$$\int_{-1}^1 f(x)dx \approx Q[f] = \sum_{i=0}^n w_i f(x_i),$$

where the $n+1$ nodes $\{x_i\}_{i=0}^n$ and weights $\{w_i\}_{i=0}^n$ are to be determined.

(a) [5] What is the minimum n to guarantee $Q[f]$ is exact for all $f \in \mathbb{P}_9$.

(b) [15] Take $n = 2$. Find the nodes $\{x_i\}_{i=0}^2$ and weights $\{w_i\}_{i=0}^2$ of the quadrature rule that maximizes the algebraic accuracy.

Hint: You can make use of Legendre polynomials $\{P_i\}_{i=0}^\infty$, which are orthogonal with respect to standard L^2 inner product:

$$\int_{-1}^1 P_m(x)P_n(x)dx = \begin{cases} 0 & m \neq n \\ \frac{2}{2n+1} & m = n \end{cases}.$$

They can be constructed recursively as $P_{n+1}(x) = \frac{2n+1}{n+1} x P_n(x) - \frac{n}{n+1} P_{n-1}(x)$, for $n = 1, 2, \cdots$, with $P_0(x) = 1$ and $P_1(x) = x$.

9. (35 points) Consider the following scheme which solves the ODE $y' = f(x,y)$.

$$y_{n+1} = y_n + \frac{h}{2}\left[f\left(x_n + \frac{h}{2}, y_n + \frac{h}{2} f(x_n, y_n) \right) + f\left(x_n + \frac{h}{2}, y_{n+1} - \frac{h}{2} f(x_{n+1}, y_{n+1}) \right) \right].$$

(a) [5] Is the scheme explicit or implicit?

(b) [15] Express the truncation error $T_n(h)$, and find the local order of accuracy.

(c) [10] Obtain the region of absolute stability of the scheme. Is the scheme A-stable?

(d) [5] Does the method converge? What is the rate of convergence? (Just state the result. No need to prove.)

