AMSC/CMSC 460 Computational Methods

Exam 1, Thursday, February 26, 2015 **Solution**

Show all work clearly and in order, and circle your final answers. Justify your answers algebraically whenever possible. Use no books, calculators, cellphones, communication with others, scratchpaper, etc. You have 80 minutes to take this 105 point exam. If you get more than 100 points, your grade will be 100.

1. (20 points) Mark each of the following statements T (True) or F (False).
   You will get 4 points for each correct answer, -1 points for each wrong answer, and 0 point for leaving it blank.

   (a) ______ Gauss elimination is the most efficient method to solve a linear system $Ax = b$ where A is a sparse nonsingular square matrix.

   Solution: False. Other methods (e.g. iterative methods like Jacobi, Gauss-Seidel etc, Thomas algorithm) are more efficient if A is sparse.

   (b) ______ Matlab script `cond(A,1)` provides the condition number of matrix A. It represents how much the absolute error of a vector is amplified by solving the linear system.

   Solution: False. Should be relative error.

   (c) ______ Suppose A is an m-by-n matrix with $m > n$. Then, there must be no solution for $Ax = b$.

   Solution: False. A might be degenerate. An example for $m = 2, n = 1$:

   $$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} x \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} \quad \Rightarrow \quad x = 2.$$

   (d) ______ Suppose Q is an n-by-n orthonormal matrix. Then $Q^TQ = \mathbb{I}$ and $QQ^T = \mathbb{I}$. Here, $\mathbb{I}$ is the n-by-n identity matrix.

   Solution: True. By definition, $Q^TQ = \mathbb{I}$. Since Q is square matrix, $Q^T = Q^{-1}$. Therefore, $QQ^T = \mathbb{I}$ as well.

   (e) ______ Secant method has a superlinear convergence rate for all initials.

   Solution: False. The convergence is local, namely initials has to be close enough to the root in order to guarantee convergence.

2. (20 points) Consider the following Matlab code.

```
A = [.5 4 .5; 1 2 -1;-.2 .8 2.6];
b = [1.5 -1 5]';

c = norm(A,inf)
[L,U,p] = lu(A,'vector')
x = A\b
```

Find the output c, L, U, p, x by hand.

Solution: For $c = \|A\|_\infty$, we have

$$\|A\|_\infty = \max\{5, 4, 3.6\} = 5.$$

For L, U, p, Gauss elimination procedure is omitted here. The result is:

$$L = \begin{pmatrix} 1 & 0 & 0 \\ .5 & 1 & 0 \\ -.2 & .4 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 0 & 0 & 2 \end{pmatrix}, \quad p = \begin{pmatrix} 2 & 1 & 3 \end{pmatrix}, \quad x = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}.$$

3. (a) (12 points) Prove that the two vector norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent. Namely, prove that for every vector $v \in \mathbb{R}^n$,

$$\|v\|_2 \le \|v\|_1 \le \sqrt{n}\|v\|_2.$$

Hint: You can apply the following Cauchy-Schwarz inequality on $a = v$ and $b = sgn(v)$ for the right inequality.

$$\left|\sum_{i=1}^n a_i b_i\right| \le \|a\|_2\|b\|_2, \quad \forall a, b \in \mathbb{R}^n.$$

Solution: For the left inequality,

$$\|v\|_2^2 = \sum_{i=1}^n |v_i|^2 \le \max_i |v_i| \sum_{i=1}^n |v_i| \le \|v\|_\infty \|v\|_1 \le \|v\|_1^2.$$

It implies $\|v\|_2 \le \|v\|_1$.
For the right inequality, we have

$$\|v\|_1 = \sum_{i=1}^n |v_i| = \sum_{i=1}^n v_i sgn(v_i) \le \|v\|_2 \|sgn(v)\|_2.$$

Note that we have used Cauchy-Schwarz inequality here. Next, we observe

$$\|sgn(v)\|_2^2 = \sum_{i=1}^{n} |sgn(v_i)|^2 \le n,$$

Since $|sgn(v_i)| \le 1$. Therefore $\|sgn(v)\|_2 \le \sqrt{n}$ and $\|v\|_1 \le \sqrt{n}\|v\|_2$.

(b) (8 points) Prove that the two matrix norms $\|\cdot\|_1$ and $\|\cdot\|_2$ induced from the corresponding vector norms are equivalent. Namely, prove that for every n-by-n matrix A,

$$\frac{1}{\sqrt{n}}\|A\|_2 \le \|A\|_1 \le \sqrt{n}\|A\|_2.$$

Solution: For the right inequality, take any $v \in \mathbb{R}^n$ such that $v \neq 0$.

$$\frac{\|Av\|_1}{\|v\|_1} \overset{\text{Use }(a)}{\le} \frac{\sqrt{n}\|Av\|_2}{\|v\|_2} \le \sqrt{n}\|A\|_2.$$

Take the superum for the left hand side, we conclude $\|A\|_1 \le \sqrt{n}\|A\|_2$.
The other inequality $\|A\|_2 \le \sqrt{n}\|A\|_1$ can be obtained similarly.

4. (a) (10 points) Find a QR decomposition of the following matrix

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 8 & -1 \\ 0 & 2 & 3 \\ -2 & 0 & 4 \end{pmatrix}.$$

Solution: Use Gram-Schmidt (omit the details) and get

$$Q = \begin{pmatrix} 1/3 & 0 & 2/3 \\ 2/3 & 2/3 & -1/3 \\ 0 & 1/3 & 2/3 \\ -2/3 & 2/3 & 0 \end{pmatrix}, \quad R = \begin{pmatrix} 3 & 6 & -3 \\ 0 & 6 & 3 \\ 0 & 0 & 3 \end{pmatrix}.$$

(b) (10 points) Use the decomposition in (a) to find a vector $x \in \mathbb{R}^3$ which minimize $\|Ax - b\|_2$ where $b = [1, 1, -1, -1]^T$.

Solution: Solve the linear system $Rx = Q^T b$, we get $x = \begin{pmatrix} 4/9 \\ 0 \\ -1/9 \end{pmatrix}$.

5. The following root finding method is called *relaxation method*. The iterative procedure is given by

$$x_{k+1} = x_k - \lambda f(x_k),$$

where λ is a constant to be determined.

(a) (5 points) Check the consistency condition: if x_k converges, then the limit must be a root of f.

Solution: We check $g(x_*) = x_* - \lambda f(x_*) = x_*$.

(b) (15 points) Suppose f is a smooth function, and $f'(x) \in [1, 2]$. Prove that the scheme converges for $0 < \lambda < 1$. What is the rate of convergence?

Solution: We check $g'(x_*) = 1 - \lambda f(x_*)$. When $\lambda \in (0, 1)$ and $f(x_*) \in [1, 2]$, we get $\lambda f(x_*) \in (0, 2)$ and hence $g'(x_*) \in (-1, 1)$. As $|g'(x_*)| < 1$, the scheme converges linearly, namely the rate of convergence is 1.

(c) (5 points) If we allow $\lambda = \lambda(x_k)$ that depends on x_k, find an expression of $\lambda(x_k)$ so that the new scheme has a higher rate of convergence. (You should be familiar with the new method.)

Solution: As $g'(x) = 1 - \lambda f'(x)$. To obtain higher convergence rate, we can pick $\lambda = \lambda(x) = \frac{1}{f'(x)}$ such that $g(x) = 0$. The new scheme converges (at least) quadraticly for smooth f. It reads

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}.$$

It is Newton's method.