# AMSC/CMSC 460 Computational Methods

**Exam 3, Thursday, December 4, 2014** **Solution**

Show all work clearly and in order, and circle your final answers. Justify your answers algebraically whenever possible. Use no books, calculators, cellphones, communication with others, etc, except a formula sheet (A4 one-sided) prepared by yourself. You have 80 minutes to take this 105 point exam. If you get more than 100 points, your grade will be 100.

1. (20 points) Mark each of the following statements T (True) or F (False).
You will get 4 points for each correct answer, -1 points for each wrong answer, and 0 point for leaving it blank.

   (a) ______ The formula $y_{n+1} = y_n + hf_{n+1}$ represents an implicit method for solving ODEs.

   **Solution:** True. $f_{n+1} = f(x_{n+1}, y_{n+1})$, which is implicit.

   (b) ______ Matlab function `ode45` should not be used for stiff systems of differential equations.

   **Solution:** True. For stiff systems, use e.g. `ode23t`.

   (c) ______ One can design a linear implicit third order scheme for solving ODEs which is A-stable.

   **Solution:** False. The maximum is second order.

   (d) ______ Suppose a multistep scheme has a truncation error of order $s$, and it is stable. Then, any algorithm with this scheme has convergence with order $s$.

   **Solution:** False. Initial steps might change the order of convergence.

   (e) ______ The projection of a vector $u$ onto $v$ can be calculated by $\frac{\langle u, v\rangle}{\|u\|^2}v$, where $\langle \cdot, \cdot \rangle$ denotes the inner product.

   **Solution:** False. Should be $\frac{< u, v >}{\|v\|^2}v$

2. (20 points) The following Matlab code solves the ODE $y' = f(x, y)$.

```
for i = 1:N+1
    k1 = f(x(i),y(i));
```

```
    k2 = f(x(i)+3/4*h,y(i)+3/4*h*k1);
    y(i+1) = y(i)+h/3*(k1+2*k2);
end
```

Write down the scheme, and find the order of accuracy.

**Solution:** The scheme is RK2. Refer to your lecture notes for details.

3. (a) (10 points) A trapezoid rule method is given as

$$y_{n+1} = y_n + \frac{h}{2}[f(x_n, y_n) + f(x_{n+1}, y_{n+1})].$$

Write down the trancation error, and prove that the method has second order accuracy.

*Hint: You are free to use the error estimate for trapezoid rule:*

$$\left| \int_a^b f(x)dx - \frac{b-a}{2}(f(a) + f(b)) \right| \leq \frac{(b-a)^3}{12} \max_{\xi \in [a,b]} |f''(\xi)|.$$

**Solution:** For the exact solution, we have

$$y(x_{n+1}) - y_n = \int_{x_n}^{x_{n+1}} f(x, y(x))dx = \frac{h}{2}(f(x_n, y_n) + f(x_{n+1}, y(x_{n+1)})) + E,$$

where $E$ is the error which is bounded by

$$\frac{h^3}{12} \max_{x \in [x_n, x_{n+1}]} \left| \frac{d^2}{dx^2} f(x, y(x)) \right|, \quad i.e. \quad \frac{h^3}{12} \max_{x \in [x_n, x_{n+1}]} |y'''(x)|.$$

Therefore, the truncation error is

$$\begin{aligned}
T_n =& \frac{y(x_{n+1}) - y_{n+1}}{h} = \frac{y(x_{n+1}) - y_n}{h} - \frac{y_{n+1} - y_n}{h} \\
=& \frac{1}{2} f(x_{n+1}, y(x_{n+1})) - \frac{1}{2} f(x_{n+1}, y_{n+1}) + \mathcal{O}(h^2) \\
(*) \quad =& \frac{1}{2} \partial_y f(x_{n+1}, y(\xi))(y(x_{n+1}) - y_{n+1}) + \mathcal{O}(h^2) \\
=& \frac{1}{2} \partial_y f(x_{n+1}, y(\xi)) h T_n + \mathcal{O}(h^2) \\
\Rightarrow \quad T_n =& \left(1 - \frac{1}{2} \partial_y f(x_{n+1}, y(\xi)) h\right)^{-1} \mathcal{O}(h^2) = \mathcal{O}(h^2).
\end{aligned}$$

Therefore, the method has second order accuracy.

Remark: Some books define truncation error for implicit method differently as follows

$$T_n = \frac{y(x_{n+1}) - y_n}{h} - \frac{h}{2}[f(x_n, y_n) + f(x_{n+1}, y(x_{n+1}))],$$

where they do not consider the term in (*) as part of the truncation error, but absorbed this part into the stability estimate.

(b) (5 points) Consider the initial value problem

$$\begin{cases} y' = -\lambda y \\ y(0) = y_0. \end{cases}$$

Write the trapezoid rule explicitly.

**Solution:** Plug in $f(x, y) = -\lambda y$ to the scheme in (a), we get

$$y_{n+1} = y_n + \frac{h}{2}[-\lambda y_n - \lambda y_{n+1}].$$

Reorganize the scheme in the explicit way:

$$\left(1 + \frac{\lambda h}{2}\right) y_{n+1} = \left(1 - \frac{\lambda h}{2}\right) y_n, \quad \Rightarrow \quad y_{n+1} = \frac{2 - \lambda h}{2 + \lambda h} y_n.$$

(c) (10 points) For $\lambda > 0$, the exact solution of the initial value problem is $y(x) = y_0 e^{-\lambda x}$, which decays as $x$ becomes larger. Prove that the method is A-stable, namely, for any stepsize $h > 0$, $|y_{n+1}| < |y_n|$.

**Solution:** Given $\lambda h > 0$, we get

$$|y_{n+1}| = \left|\frac{2 - \lambda h}{2 + \lambda h}\right| |y_n| < |y_n|.$$

Therefore, the method is A-stable.

(d) (5 points) Does the method converge? What is the rate of convergence? (Just state the result. No need to prove.)

**Solution:** The method has second order convergence, as it is accurate with order 2 and stable.

4. Consider the following linear system:

$$\begin{aligned} \alpha - \beta &= 1 \\ -\alpha + 2\beta &= 2 \\ 2\alpha + \beta &= 1 \end{aligned}$$

(a) (5 points) Write the system in matrix form $Ax = b$. Does this system have a solution?

**Solution:** The system can be rewritten as

$$\begin{pmatrix} 1 & -1 \\ -1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}.$$

It has NO solution as the augmented matrix has rank 3 which is more than the number of unknowns 2.

(b) (10 points) Find $x$ that minimizes the error in $L^2$, namely, $\min_x \|Ax - b\|_2^2$.

**Solution:** The minimizer is given by $x = (A^TA)^{-1}A^Tb$. Plug in the specific $A$ and $b$, we get $x = [\alpha, \beta]^T = [2/7, 5/7]^T$.

5. (20 points) Find a QR docomposition of the following matrix

$$A = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 3 & 1 \\ 1 & 3 & 3 \\ 1 & 1 & 5 \end{pmatrix}.$$

**Solution:** Use Gram-Schmidt (omit the details) and get

$$Q = \begin{pmatrix} 1/2 & -1/2 & -1/2 \\ 1/2 & 1/2 & -1/2 \\ 1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & 1/2 \end{pmatrix}, \quad R = \begin{pmatrix} 2 & 4 & 6 \\ 0 & 2 & -2 \\ 0 & 0 & 2 \end{pmatrix}.$$