

AMSC/CMSC 460 Computational Methods

Exam 2, Thursday, November 6, 2014

Show all work clearly and in order, and circle your final answers. Justify your answers algebraically whenever possible. Use no books, calculators, cellphones, communication with others, etc, except a formula sheet (A4 one-sided) prepared by yourself. You have 80 minutes to take this 105 point exam. If you get more than 100 points, your grade will be 100.

1. (20 points) Mark each of the following statements T (True) or F (False).

You will get 4 points for each correct answer, -1 points for each wrong answer, and 0 point for leaving it blank.

- (a) _____ There exists a unique polynomial of degree up to 5 that interpolates a function at 6 distinct nodes.
- (b) _____ A Hermite cubic spline is a C^2 function.
- (c) _____ Suppose $s(x)$ is a piecewise linear polynomial with respect to knots $\{x_i\}_{i=0}^m$, which minimizes $\|f - s\|_2$. Then, s interpolates f at the knots.
- (d) _____ Matlab script `sparse([1, 2], [3, 1], [1, 2])` generates a sparse matrix $\begin{pmatrix} 0 & 0 & 1 \\ 2 & 0 & 0 \end{pmatrix}$.
- (e) _____ Gaussian integration uses equally spaced points.
2. $p_2(x)$ is a quadratic polynomial which interpolates $f(x) = (1 + |x|)^{-1/2}$ at three nodes $x = -3, 0, 3$.
- (a) (10 points) Express $p_2(x)$ in the form of $ax^2 + bx + c$.
- (b) (10 points) Obtain an error bound uniformly in $[-3, 3]$. Namely, find an upper bound of $\|f - p_2\|_{L^\infty([-3, 3])}$.
3. (20 points) Find the cubic polynomial $p_3(x)$ which minimizes $\|f - p_3\|_{L^2([-1, 1])}$, for function $f(x) = x^4$.

Hint: You might make use of Legendre polynomials $\{P_n(x)\}_{n=0}^\infty$, which are orthogonal with respect to L^2 norm.

$$\int_{-1}^1 P_m(x)P_n(x)dx = \begin{cases} 0 & m \neq n \\ \frac{2}{2m+1} & m = n \end{cases}.$$

The first 4 Legendre polynomials is given as below:

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{2}(3x^2 - 1), \quad P_3(x) = \frac{1}{2}(5x^3 - 3x).$$

To save you some computational load, $\int_{-1}^1 x^4 P_2(x)dx = \frac{8}{35}$ and $\int_{-1}^1 x^4 P_3(x)dx = 0$.

4. (a) (5 points) Write a composite trapezoid rule on $\int_0^1 f(x)dx$ using 6 equally distributed nodes $\{x_i\}_{i=0}^5$, where $x_i = i/5$.
- (b) (10 points) Suppose $\max_{0 \leq x \leq 1} |f''(x)| = 1$. Give an upper bound on the error.
Hint: You can use the following error estimate on trapezoid rule without a proof.

$$\left| \int_a^b f(x)dx - \frac{b-a}{2}[f(a) + f(b)] \right| \leq \frac{(b-a)^3}{12} \max_{x \in [a,b]} |f''(x)|.$$

5. Chebyshev polynomials $\{T_n\}_{n=0}^\infty$ are orthogonal with respect to the weight $w(x) = \frac{1}{\sqrt{1-x^2}}$ on $[-1, 1]$. More precisely, they satisfy the following identities.

$$\int_{-1}^1 T_m(x)T_n(x) \frac{1}{\sqrt{1-x^2}} dx = \begin{cases} 0 & m \neq n \\ \pi & m = n = 0 \\ \frac{\pi}{2} & m = n \neq 0 \end{cases}.$$

Chebyshev polynomials can be defined recursively by

$$T_0(x) = 1, \quad T_1(x) = x, \quad \text{and} \quad T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \quad \text{for } n \geq 1.$$

- (a) (5 points) Find the first five Chebyshev polynomials $\{T_n\}_{n=0}^4$.
- (b) (15 points) Set up a Gauss quadrature for the following numerical integration.

$$\int_{-1}^1 f(x) \frac{1}{\sqrt{1-x^2}} dx \approx w_0 f(x_0) + w_1 f(x_1) + w_2 f(x_2).$$

Find the nodes $\{x_i\}_{i=0}^2$ and weights $\{w_i\}_{i=0}^2$ such that the approximation is precise for all $f \in \mathbb{P}_5$.

- (c) (10 points) Use the quadrature rule to find
- the exact value of the integral $\int_{-1}^1 \frac{x^5 - 1}{\sqrt{1-x^2}} dx$,
 - an approximate value of the integral $\int_{-1}^1 \sqrt{\frac{4x^2 + 1}{1-x^2}} dx$.