AMSC/CMSC 460 Computational Methods

Exam 1, Tuesday, September 30, 2014 **Solution**

1. (20 points) Mark each of the following statements T (True) or F (False).
You will get 4 points for each correct answer, -2 points for each wrong answer, and 0 point for leaving it blank.

   (a) ______ Suppose A is an n-by-n square matrix. b is any vector in $\mathbb{R}^n$. Then, $Ax = b$ has a unique solution $x \in \mathbb{R}^n$.

   > **Solution:** True. There exists a unique solution for an n-by-n non-degenerate linear system.

   (b) ______ $\|A\|_\infty = \|A^T\|_1$, where $\|\cdot\|_\infty$ and $\|\cdot\|_1$ are matrix norms induced from corresponding vector norms.

   > **Solution:** True. This is a direct consequence from the definations of the two norms.

   (c) ______ Let E_{ij} be the matrix with all zero entries expect the (i, j)-entry being 1. Then $E_{i_1j_1}$ commutes with $E_{i_2j_2}$ if $i_1 > j_1$ and $i_2 > j_2$.

   > **Solution:** False. If $j_1 = i_2$, then $E_{i_1j_1}E_{i_2j_2} = E_{i_1j_2}$ while $E_{i_2j_2}E_{i_1j_1} = 0$.

   (d) ______ Matlab code `A*(B*v)` runs faster than `(A*B)*v`. Here, `A`, `B` are matrices and `v` is a vector.

   > **Solution:** True. Computing `A*B` requires matrix-matrix multiplication, which is slower than matrix-vector multiplication.

   (e) ______ Newton's method always have a local quadratic convergnce.

   > **Solution:** False. If the root is not a single root, then the convergence is linear.

2. (20 points) Consider the following Matlab code.

```
A = [1 2 3;3 2 1;2 0 2];
b = [-5; -2; 5];
[L,U,P] = lu(A)
[L2,U2] = lu(A)
x = A\b
```

Find the output L, U, P, L2, U2, x by hand using Gaussian elimination.

Solution: Gauss elimination procedure is omitted here. The result is:

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 1/3 & 1 & 0 \\ 2/3 & -1 & 1 \end{pmatrix}, \quad U = U2 = \begin{pmatrix} 3 & 2 & 1 \\ 0 & 4/3 & 8/3 \\ 0 & 0 & 4 \end{pmatrix},$$

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad L2 = P^{-1}L = \begin{pmatrix} 1/3 & 1 & 0 \\ 1 & 0/ & 0 \\ 2/3 & -1 & 1 \end{pmatrix}, \quad x = \begin{pmatrix} 2 \\ -17/4 \\ 1/2 \end{pmatrix}.$$

3. (a) (10 points) Prove that the two vector norms $\|\cdot\|_1$ and $\|\cdot\|_\infty$ are equivalent. Namely, prove that for every vector $v \in \mathbb{R}^n$,

$$\|v\|_\infty \leq \|v\|_1 \leq n\|v\|_\infty.$$

Solution: For the right inequality, we have

$$\|v\|_1 = \sum_{i=1}^{n} |v_i| \leq \sum_{i=1}^{n} \max_k |v_k| = n\|v\|_\infty.$$

For the left inequality, suppose k^* is the entry that $|v_{k^*}| = \max_k |v_k| = \|v\|_\infty$. Then,

$$\|v\|_1 = |v_{k^*}| + \sum_{k \neq k^*} |v_k| \geq |v_{k^*}| + 0 = \|v\|_\infty.$$

(b) (10 points) Prove that the two matrix norms $\|\cdot\|_1$ and $\|\cdot\|_\infty$ induced from the corresponding vector norms are equivalent. Namely, prove that for every n-by-n matrix A,

$$\frac{1}{n}\|A\|_\infty \leq \|A\|_1 \leq n\|A\|_\infty.$$

Solution: For the right inequality, take any $v \in \mathbb{R}^n$ such that $v \neq 0$.

$$\frac{\|Av\|_1}{\|v\|_1} \overset{\text{Use (a)}}{\leq} \frac{n\|Av\|_\infty}{\|v\|_\infty} \leq n\|A\|_\infty.$$

Take the superum for the left hand side, we conclude $\|A\|_1 \leq n\|A\|_\infty$.
The other inequality $\|A\|_\infty \leq n\|A\|_1$ can be obtained similarly.

4. Consider the following Matlab code on finding a root of $f(x)$.

```
k = 0;
while abs(x-xprev) > eps*abs(x)
    xprev = x;
    x = x - f(x)^2/(f(x+f(x))-f(x));
    k = k + 1;
end
```

(a) (5 points) Write down the iterative procedure in the form $x_{k+1} = g(x_k)$.

Solution: The iterative procedure is

$$x_{k+1} = x_k - \frac{f(x_k)^2}{f(x_k + f(x_k)) - f(x_k)}, \quad g(x) = x - \frac{f(x)^2}{f(x + f(x)) - f(x)}.$$

(b) (20 points) Assume x_* is a root of a smooth function $f \in C^\infty$ with $f'(x_*) \neq 0$ and $f''(x_*) \neq 0$. Prove that the method has a local convergence. What is the convergence rate α? Compute the leading coefficient $\lim_{k\to\infty} \frac{e_{k+1}}{e_k^\alpha}$.

Hint: Prepare for heavy computation. Following the steps below might lower your computational burden. (Note: you can try alternative ways, e.g. compute $g(x_*)$, $g'(x_*)$ and $g''(x_*)$ directly, but keep in mind it would cost a lot of time.)

- *Take $F(x) = \frac{f(x + f(x)) - f(x)}{f(x)}$ and express g in terms of f and F.*
- *Taylor expand $f(x + f(x))$ around x and write F as a infinite series. Check that F has the following form:*

$$F(x) = \sum_{n=0}^{\infty} \frac{f^{(n+1)}(x)}{(n+1)!} f(x)^n.$$

- *Compute $F'(x), F''(x)$, and evaluate F, F', F'' at $x = x_*$. Note that $f(x_*) = 0$.*
- *Given $F(x_*), F'(x_*), F''(x_*)$, compute $g(x_*), g'(x_*)$ and $g''(x_*)$. The local convergence is given by checking $g(x_*) = x_*$.*
- *Conclude with the convergence rate α and leading coefficient $\lim_{k\to\infty} \frac{e_{k+1}}{e_k^\alpha}$.*

Solution: We first calculate $F(x), F'(x)$ and $F''(x)$. As f is smooth, we can write F by Taylor-expand $f(x+f(x))$ around x:

$$F(x)=\frac{1}{f(x)}\left[\sum_{n=0}^{\infty}\frac{f^{(n)}(x)}{n!}f(x)^n-f(x)\right]=\sum_{n=0}^{\infty}\frac{f^{(n+1)}(x)}{(n+1)!}f(x)^n.$$

$$F'(x)=\sum_{n=0}^{\infty}\frac{f^{(n+2)}(x)}{(n+1)!}f(x)^n+\sum_{n=1}^{\infty}\frac{f^{(n+1)}(x)}{(n+1)!}nf(x)^{n-1}f'(x)$$

$$F''(x)=\sum_{n=0}^{\infty}\frac{f^{(n+3)}(x)}{(n+1)!}f(x)^n+2\sum_{n=1}^{\infty}\frac{f^{(n+2)}(x)}{(n+1)!}nf(x)^{n-1}f'(x)$$
$$+\sum_{n=2}^{\infty}\frac{f^{(n+1)}(x)}{(n+1)!}n[(n-1)f(x)^{n-2}f'(x)+f(x)^{n-1}f''(x)].$$

To evaluate $F(x),\cdots$ at x_*, note that all $f(x_*)^n=0$ unless $n=0$. Hence, only zero-th order term left.

$$F(x_*)=f'(x_*)$$
$$F'(x_*)=f''(x_*)+\frac{f''(x_*)}{2}f'(x_*)=\frac{f''(x_*)(2+f'(x_*))}{2}$$
$$F''(x)=f'''(x_*)+f'''(x_*)f'(x_*)+\frac{f'''(x_*)}{3}f'(x_*)=\frac{f'''(x_*)(3+4f'(x_*))}{3}.$$

Next, we compute g,g',g'' in terms of f and F.

$$g(x)=x-\frac{f(x)}{F(x)}$$
$$g'(x)=1-\frac{f'(x)}{F(x)}+\frac{f(x)F'(x)}{F(x)^2}$$
$$g''(x)=-\frac{f''(x)}{F(x)}+\frac{2f'(x)F'(x)}{F(x)^2}-f(x)\left(\frac{1}{F(x)}\right)''$$

Clearly, $\left(\frac{1}{F(x_*)}\right)''$ is bounded as $F'(x_*)\neq 0$ and F,F',F'' are bounded at x_* as well. Hence, let $x=x_*$, we obtain

$$g(x_*)=x_*-\frac{f(x_*)}{f'(x_*)}=x_*$$
$$g'(x_*)=1-\frac{f'(x_*)}{f'(x_*)}=0$$
$$g''(x_*)=-\frac{f''(x_*)}{f'(x_*)}+\frac{2f'(x_*)}{f'(x_*)^2}\cdot\frac{f''(x_*)(2+f'(x_*))}{2}-0=\frac{f''(x_*)(1+f'(x_*))}{f'(x_*)}.$$

To conclude, the method has a local convergence rate $\alpha=2$. And the leading coefficient

$$\lim_{k\to\infty}\frac{|e_{k+1}|}{|e_k|^2}=\frac{|f''(x_*)(1+f'(x_*))|}{2|f'(x_*)|}.$$

An alternative way to check $g(x_*) = x_*$:

$$\begin{aligned} g(x_*) = &\ x_* - \lim_{x\to x^*} \frac{f(x)^2}{f(x+f(x)) - f(x)} \\ \overset{"0/0"}{=} &\ x_* - \lim_{x\to x^*} \frac{2f(x)f'(x)}{f'(x+f(x))(1+f'(x)) - f'(x)} \\ = &\ x_* - \frac{f(x_*)f'(x_*)}{f'(x_*)^2} = x_*. \end{aligned}$$

But note that to check $g'(x_*) = 0$ is much more complicated under similar procedure.

5. (20 points) Write a Newton's iterative scheme to solve the following nonlinear system.

$$\begin{cases} x^2 + e^{xy} = 6 \\ x - y^2 = 2 \end{cases}.$$

If we take initially $x = 2, y = 0$, what is the value of (x, y) after 1 iteration?

Solution: The Newton's iterative method reads

$$\mathbf{x}_{k+1} = \mathbf{x}_k - J(\mathbf{x}_k)^{-1}\mathbf{f}(\mathbf{x}_k),$$

where $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$, $\mathbf{f}(\mathbf{x}) = \begin{pmatrix} x^2 + e^{xy} - 6 \\ x - y^2 - 2 \end{pmatrix}$ and $J(\mathbf{x}) = \nabla_{\mathbf{x}}\mathbf{f} = \begin{pmatrix} 2x + ye^{xy} & xe^{xy} \\ 1 & -2y \end{pmatrix}$.

In particular, if $\mathbf{x}_0 = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$, we get $\mathbf{f}(\mathbf{x}_0) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$ and $J(\mathbf{x}_0) = \nabla_{\mathbf{x}}\mathbf{f} = \begin{pmatrix} 4 & 2 \\ 1 & 0 \end{pmatrix}$.

By solving the linear system $\begin{pmatrix} 4 & 2 \\ 1 & 0 \end{pmatrix} \mathbf{s_0} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$ and get $\mathbf{s_0} = \begin{pmatrix} 0 \\ -.5 \end{pmatrix}$. Therefore, $\mathbf{x}_1 = \mathbf{x}_0 - \mathbf{s_0} = \begin{pmatrix} 2 \\ .5 \end{pmatrix}$.