AMSC/CMSC 460 Computational Methods

Exam 1, Tuesday, September 30, 2014

Show all work clearly and in order, and circle your final answers. Justify your answers algebraically whenever possible. Use no books, calculators, cellphones, communication with others, scratchpaper, etc. You have 80 minutes to take this 105 point exam. If you get more than 100 points, your grade will be 100.

1. (20 points) Mark each of the following statements T (True) or F (False).
   You will get 4 points for each correct answer, -2 points for each wrong answer, and 0 point for leaving it blank.
   (a) ______ Suppose A is an n-by-n square matrix with $det(A) \neq 0$. b is any vector in $\mathbb{R}^n$. Then, $Ax = b$ has a unique solution $x \in \mathbb{R}^n$.
   (b) ______ $\|A\|_\infty = \|A^T\|_1$, where $\|\cdot\|_\infty$ and $\|\cdot\|_1$ are matrix norms induced from corresponding vector norms.
   (c) ______ Let E_{ij} be the matrix with all zero entries expect the (i,j)-entry being 1. Then $E_{i_1 j_1}$ commutes with $E_{i_2 j_2}$ if $i_1 > j_1$ and $i_2 > j_2$.
   (d) ______ Matlab code `A*(B*v)` runs faster than `(A*B)*v`. Here, `A`, `B` are matrices and `v` is a vector.
   (e) ______ Newton's method always have a local quadratic convergnce.

2. (20 points) Consider the following Matlab code.

```
A = [1 2 3;3 2 1;2 0 2];
b = [-5; -2; 5];
[L,U,P] = lu(A)
[L2,U2] = lu(A)
x = A\b
```

   Find the output `L`, `U`, `P`, `L2`, `U2`, `x` by hand using Gaussian elimination.

3. (a) (10 points) Prove that the two vector norms $\|\cdot\|_1$ and $\|\cdot\|_\infty$ are equivalent. Namely, prove that for every vector $v \in \mathbb{R}^n$,

$$\|v\|_\infty \leq \|v\|_1 \leq n\|v\|_\infty.$$

   (b) (10 points) Prove that the two matrix norms $\|\cdot\|_1$ and $\|\cdot\|_\infty$ induced from the corresponding vector norms are equivalent. Namely, prove that for every n-by-n matrix A,

$$\frac{1}{n}\|A\|_\infty \leq \|A\|_1 \leq n\|A\|_\infty.$$

© 2014 Changhui Tan

4. Consider the following Matlab code on finding a root of $f(x)$.

```
k = 0;
while abs(x-xprev) > eps*abs(x)
    xprev = x;
    x = x - f(x)^2/(f(x+f(x))-f(x));
    k = k + 1;
end
```

(a) (5 points) Write down the iterative procedure in the form $x_{k+1} = g(x_k)$.

(b) (20 points) Assume x_* is a root of a smooth function $f \in C^\infty$ with $f'(x_*) \neq 0$ and $f''(x_*) \neq 0$. Prove that the method has a local convergence. What is the convergence rate α? Compute the leading coefficient $\lim_{k\to\infty} \frac{e_{k+1}}{e_k^\alpha}$.

Hint: Prepare for heavy computation. Following the steps below might lower your computational burden. (Note: you can try alternative ways, e.g. compute $g(x_*), g'(x_*)$ and $g''(x_*)$ directly, but keep in mind it would cost a lot of time.)

- *Take $F(x) = \frac{f(x+f(x)) - f(x)}{f(x)}$ and express g in terms of f and F.*
- *Taylor expand $f(x+f(x))$ around x and write F as a infinite series. Check that F has the following form:*

$$F(x) = \sum_{n=0}^{\infty} \frac{f^{(n+1)}(x)}{(n+1)!} f(x)^n.$$

- *Compute $F'(x), F''(x)$, and evaluate F, F', F'' at $x = x_*$. Note that $f(x_*) = 0$.*
- *Given $F(x_*), F'(x_*), F''(x_*)$, compute $g(x_*), g'(x_*)$ and $g''(x_*)$. The local convergence is given by checking $g(x_*) = x_*$.*
- *Conclude with the convergence rate α and leading coefficient $\lim_{k\to\infty} \frac{e_{k+1}}{e_k^\alpha}$.*

5. (20 points) Write a Newton's iterative scheme to solve the following nonlinear system.

$$\begin{cases} x^2 + e^{xy} = 6 \\ x - y^2 = 2 \end{cases}.$$

If we take initially $x = 2, y = 0$, what is the value of (x, y) after 1 iteration?

© 2014 Changhui Tan