CAAM 423/523 Partial Differential Equations I
MATH 423/513 Fall 2017

Homework 9, Due on Tuesday, November 28, 2017

1.(*) *(Poisson's formula on a half ball) Required for MATH513/CAAM523 students only.*

a). Read the derivation of Poisson's formula for ball (page 39-41 in Evans book).

b). Consider half ball $U^+ := \{x \in \mathbb{R}^n \ : \ |x| < 1, \ x_n > 0\}$. The boundary value problem states

$$\begin{cases} \Delta u = 0 & \text{in } U^+ \\ u = g_1 & \text{on } \partial B_1(0) \cap \{x_n > 0\} \\ u = g_2 & \text{on } B_1(0) \cap \{x_n = 0\} \end{cases}$$

where the two boundary values g_1 and g_2 are set to be compatible when they intersect.

Verify that a Green's function can be expressed as

$$G(x, y) := \Phi(y - x) - \Phi(y - \hat{x}) - \Phi(|x|(y - \tilde{x})) + \Phi(|x|(y - \tilde{\hat{x}})),$$

where $\hat{x} = (x_1, \cdots, x_{n-1}, -x_n)$, and $\tilde{x} = x/|x|^2$.

Note: one has to check the corresponding $\phi^x(y)$ satisfies the boundary value problem. Use the results for half-space and ball directly.

c). Write down the Poisson's formula for Laplace equation on U^+. (You do not need to verify that the solution indeed solves the boudary value problem. But keep in mind that the verification is necessary, and the procedure is similar to the half-space and ball cases.)

2. *(Harnack's inequality in a ball)* Finish excercise 7 in section 2.5 of Evans book. Use Poisson's formula for the ball to prove

$$r^{n-2} \frac{r - |x|}{(r + |x|)^{n-1}} u(0) \leq u(x) \leq r^{n-2} \frac{r + |x|}{(r - |x|)^{n-1}} u(0)$$

whenever u is positive and harmonic in $B_r(0)$.

3. *(Loss of smoothness for solutions for Laplace equation at the boundary)* Finish excercise 9 in section 2.5 of Evans book. Let u be the solution of

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^n_+ \\ u = g & \text{on } \partial\mathbb{R}^n_+ \end{cases}$$

given by Poisson's formula for the half space. Assume g is bounded and $g(x) = |x|$ for $x \in \partial\mathbb{R}^n_+, |x| \leq 1$. Show Du is *not* bounded near $x = 0$. *Hint: estimate $\partial_{x_n} u(x)$ at point $x = 0$.*

4. *(Solving heat equation using Fourier transform)* Consider the initial value problem for heat equation

$$\begin{cases} u_t - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0, \infty) \\ u = g & \text{on } \mathbb{R}^n \times \{t = 0\}. \end{cases}$$

a). Take Fourier transform in of the equation in x, and solve the initial value problem for $\hat{u}$.

b). *(This part is optional, as it requires calculations in the complex plane.)* Verify that $\mathcal{F}^{-1}(e^{-|y|^2 t})(x) = \frac{1}{(2t)^{n/2}} e^{-\frac{|x|^2}{4t}}$ for $t > 0$, where the inverse Fourier transform $\mathcal{F}^{-1}$ is with respect to y.

c). Derive the solution of the initial value problem based on a) and b).

Note: one still have to verify whether the initial condition is satisfied, as the Fourier transform is not valid for $t = 0$. *The procedure is the same as discussed in class. You do not have to repeat it in the homework.*