CAAM 423/523 Partial Differential Equations I
MATH 423/513

Fall 2017

Homework 3, Due on Tuesday, October 3, 2017

1. *(Non-uniqueness of weak solution)* Consider the following Riemann problem for Burgers equation

$$\begin{cases} u_t + uu_x = 0 & \text{in } \mathbb{R}\times(0,\infty) \\ u = H & \text{on } \mathbb{R}\times\{t=0\} \end{cases}, \quad \text{where} \quad H(x) = \begin{cases} 0 & \text{if } x<0 \\ 1 & \text{if } x>0 \end{cases}.$$

a). Write down the definition of the weak solution.

b). Verify that $u(x,t) = H(x-t/2)$ is a weak solution.

c). Verify that the problem has another weak solution, *rarefaction solution*:

$$u(x) = \begin{cases} 0 & \text{if } x<0 \\ \frac{x}{t} & \text{if } 0<x<t \\ 1 & \text{if } x>t \end{cases}.$$

Remark: Since the rarefaction solution is continuous, it satisfies Lax entropy condition. Hence, it is the unique entropy solution of the Riemann problem.

d). Find another weak solution of the Riemann problem. *Hint: one option could be a solution with two shocks. Each shock should satisfy Rankine-Hugoniot condition.*

2. *(Entropy solution for general initial data)* Finish Exercise 20 in section 3.5 of Evans book. Compute explicitely the unique entropy solution of

$$\begin{cases} u_t + uu_x = 0 & \text{in } \mathbb{R}\times(0,\infty) \\ u = g & \text{on } \mathbb{R}\times\{t=0\} \end{cases}$$

for

$$g(x) = \begin{cases} 1 & \text{if } x<-1 \\ 0 & \text{if } -1<x<0 \\ 2 & \text{if } 0<x<1 \\ 0 & \text{if } x>1 \end{cases}.$$

Draw a picture documenting your answer, being sure to illustrate what happens for all time $t>0$.

3. *(Explicit solutions for non-convex flux)* Consider the following Riemann problem

$$\begin{cases} u_t + (u^3-u)_x = 0 & \text{in } \mathbb{R}\times(0,\infty) \\ u = g & \text{on } \mathbb{R}\times\{t=0\} \end{cases}, \quad \text{for} \quad g(x) = \begin{cases} -1 & \text{if } x<0 \\ 1 & \text{if } x>0 \end{cases}.$$

a). Find the shock wave solution. Check that it is not an entropy solution.
Hint: You can check that Olynik condition does not hold at some point $k\in(-1,1)$.

b). Find the unique entropy solution of the initial value problem.