CAAM 423/523 Partial Differential Equations I
MATH 423/513 Fall 2017

Homework 2, Due on Tuesday, September 19, 2017

*(The problems marked with * are optional for MATH423 and CAAM423 students.)*

1. *(Fully nonlinear first order PDE)* Find a smooth solution to

$$u_t + (u_x)^4 = 0, \quad x \in \mathbb{R}, \quad t > 0,$$

with initial condition

$$u(x, t = 0) = \frac{3}{4} x^{4/3}.$$

2. *(Initial-boundary value problem)* Find the explicit solution to the following equation on $\mathbb{R}_+ \times \mathbb{R}_+$

$$u_t + u_x = 0, \quad t \in \mathbb{R}_+, \quad x \in \mathbb{R}_+,$$

subject to initial condition

$$u(x, t = 0) = \cos x,$$

and boundary condition

$$u(x = 0, t) = 1.$$

3. **(Non-existence of weak derivative)* Let H be the *Heaviside function* defined as

$$H(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}.$$

Prove that there exists no bounded function u such that $H_x = u$ in the weak sense.
Hint: We prove the statement by contradiction. Suppose there exists a bounded function u which is a weak derivative of H. Construct a sequence of smooth test functions $\{\phi_m\}_{m=1}^{\infty}$, such that $\phi_m(0) = 1$, and supp(ϕ_m) $= (-1/m, 1/m)$. Use the definition of the weak derivative to show $\int_{\mathbb{R}} u(x)\phi_m(x) = 1$ for all m. Then, prove a contradiction when m is big enough.

4. *(Dirac delta)* The Dirac delta δ is a linear functional of continuous function in $\mathbb{R}$ defined as

$$\delta[f] = \int_{\mathbb{R}} \delta(x) f(x) dx := f(0).$$

a). Verify that δ is a weak solution of the Heaviside function H.
Remark: from problem 3, we know that δ is not a bounded function.

b). A *convolution* between a function f and a distribution g is defined as

$$(f * g)(x) = \int_{\mathbb{R}} f(x - y) g(y) dy.$$

Given a function f, find $f * \delta$.

c). Write a defination of the weak derivative of δ.