CAAM 423/523 Partial Differential Equations I
MATH 423/513 **Fall 2017**

Sample Final Exam Name: ____________________

Instruction: (READ CAREFULLY!!)
Show all work clearly and in order. Textbook (Evans), your own notes and your own homework assignments are allowed during the exam. Use **no** other books, calculators, cellphones, communication with others, computers, etc. You are required to sign the *honor pledge*.

For CAAM423/MATH423 students: The exam has a total of 100 points. You are required to finish problems 1-6. Use **three** consecutive hours of your choice to finish the exam.

For CAAM523/MATH513 students: The exam has a total of 120 points. You are required to finish problems 1-7. Use **four** consecutive hours of your choice to finish the exam.

1. (20 points) Find the explicit local solution for the following initial value problem

$$\begin{cases} u_t + \frac{(u_x)^2+x^2}{2} = 0, \quad x \in \mathbb{R}, \quad t > 0, \\ u(x, t=0) = \frac{x^2}{2}. \end{cases}$$

2. (20 points) Consider the following equation

$$u_t + \left(\frac{u^2}{2} + u\right)_x = 0, \quad x \in \mathbb{R},\ t \geq 0,$$

with initial condition

$$u(x, t=0) = g(x) = \begin{cases} -2 & -1 < x < 0 \\ 0 & \text{otherwise} \end{cases}.$$

   (a) Write down the weak formulation of the initial value problem.
   (b) Find the *explicit* entropy solution of the problem.

3. (10 points) Consider the following system of scaler conservation laws on (u, v)

$$\begin{cases} u_t - (q(v))_x = 0 \\ v_t - (p(u))_x = 0, \end{cases}$$

where p, q are C^1 functions with $p', q' > 0$. Show that the system is strictly hyperbolic.

4. (15 points) Consider Airy's equation $u_t + u_{xxx} = 0$, with initial condition $u(x, 0) = g(x)$, for $x \in \mathbb{R}$.
   (a) Find the wave speed $\left|\sigma/|y|\right|$ for any wave number y. Is the equation dispersive?
   (b) Write down the definition of Fourier transform $\hat{u}(y, t)$, and solve $\hat{u}$.
   (c) Prove that the solution preserves L^2 norm in time: $\|u(\cdot, t)\|_{L^2(\mathbb{R})} = \|g\|_{L^2(\mathbb{R})}$, for $t > 0$.

5. (20 points) Find the *explicit* solution in the first quadrant $x > 0$ and $t > 0$ of the wave equation with initial-boundary conditions,

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0, & x > 0, t > 0, \\ u(x,0) = g(x), & u_t(x,0) = h(x), \\ u_t(0,t) = a u_x(0,t), & a \neq -c, \end{cases}$$

where $g(x)$ and $h(x)$ are C^2 functions which vanish near $x = 0$. Show that no solution exists in general if $a = -c$.

Hint: The solution of 1D wave equation can be written as $u = F(x+ct)+G(x-ct)$. Determine $F(x)$ and $G(x)$ for $x \geq 0$ from the initial condition, and determine $G(x)$ for $x < 0$ from the boundary condition.

6. (15 points) Consider heat equation with a source term and a Dirichlet boundary condition

$$\begin{cases} u_t - \Delta u = f & \text{in } \Omega \times (0, +\infty) \\ u(x,0) = g(x) & \text{for } x \in \Omega,\ t = 0 \\ u(x,t) = h(x,t) & \text{for } x \in \partial\Omega,\ t \in (0,\infty) \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^n$. Prove that there is at most one classical solution which solves the initial-boundary value problem.

7. (20 points) Let u be a smooth solution of

$$\begin{cases} -\Delta u = f & \text{in } B_1(0) \\ u = g & \text{on } \partial B_1(0). \end{cases},$$

where f and g are bounded functions in $B_1(0)$ and $\partial B_1(0)$, respectively.

(a) We say $v \in C^2(\bar{\Omega})$ is *subharmonic* if $-\Delta v \leq 0$. Prove the mean value theorem for subharmonic v that

$$v(x) \leq \fint_{B_r(x)} v(y)dy \quad \text{for all } B_r(x) \in \Omega.$$

(b) Prove that if v is subharmonic, then its satisfies the maximum principle

$$\max_{\bar{\Omega}} v = \max_{\partial\Omega} v.$$

(c) Let $v(x) = u(x) + \frac{M}{2n}|x|^2$, where $M = \max_{|x|\leq 1} |f(x)|$. Prove that v is subharmonic in $B_1(0)$.

(d) Prove that there exists a constant C, depending only on dimension n, such that

$$\max_{B_1(0)} |u| \leq C \left(\max_{\partial B_1(0)} |g| + \max_{B_1(0)} |f| \right).$$

Hint: apply maximum principle on v in (c).