CAAM/MATH423 Partial Differential Equations I — Fall 2015

Midterm Exam Name: ____________________

Instruction: (READ CAREFULLY!!) Show all work clearly and in order. The exam has a total of 110 points. At most 100 points will be recorded. Use **two** consecutive hours of your choice to finish the exam. You may choose to extend for one hour (so a total of *three* consecutive hours) to finish. If so, 10 points will be deducted from your total grades. **State** your starting and ending time below. Textbook (Evans), your own notes and your own homework assignments are allowed during the exam. Use **no** other books, calculators, cellphones, communication with others, computers, etc. You are required to sign the *honor pledge*. Please hand in the exam on October 20 in class. Please attach this page with your name and signature on submission.

Starting time: ____________________ Ending time: ____________________

Honor Pledge: The Rice University Honor Pledge reads:

> "On my honor, I have neither given nor received any unauthorized aid on this exam."

Please write the exact wording of the Pledge, following by your signature, in the space below:

Pledge: ____

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Your Signature: ____

1. (25 points) Consider the equation

$$-(u_{x_1})^2 + (u_{x_2})^2 + x_2^2 = 0, \quad x_1 \in \mathbb{R},\ x_2 > 0$$

with initial condition $u(x_1, 0) = g(x_1)$, where $g(x_1) \in C^1$ is strictly increasing. Also assume $u_{x_2}(x_1, 0) \geq 0$.

   (a) Find explicitly the characteristics $x_1(\alpha, s)$ and $x_2(\alpha, s)$ starting at the point $(\alpha, 0)$.

   (b) In the case when $g(x_1) = x_1$, sketch the characteristics and write down an explicit solution. *Hint: To save you some time, use directly $\int_0^t \sin^2(2s)ds = \frac{t}{2} - \frac{1}{8}\sin(4t)$. Note that you only have to run $s \in [0, \pi/4]$ to construct a solution.*

2. (20 points) Consider Airy's equation $u_t + u_{xxx} = 0$, with initial condition $u(x, 0) = g(x)$, for $x \in \mathbb{R}$.

   (a) Find the wave speed $|\sigma/|y||$ for any wave number y. Is the equation dispersive?

(b) Write down the definition of Fourier transform $\hat{u}(y,t)$, and solve $\hat{u}$.

(c) Prove that the solution preserves L^2 norm in time: $\|u(\cdot,t)\|_{L^2(\mathbb{R})} = \|g\|_{L^2(\mathbb{R})}$, for $t > 0$.

3. (25 points) Find the *explicit* solution in the first quadrant $x > 0$ and $t > 0$ of the wave equation with initial-boundary conditions,

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0, \quad x > 0, t > 0, \\ u(x,0) = g(x), \quad u_t(x,0) = h(x), \\ u_t(0,t) = a u_x(0,t), \quad a \neq -c, \end{cases}$$

where $f(x)$ and $g(x)$ are C^2 functions which vanish near $x = 0$. Show that no solution exists in general if $a = -c$.

Hint: The solution of 1D wave equation can be written as $u = F(x+ct) + G(x-ct)$. Determine $F(x)$ and $G(x)$ for $x \geq 0$ from the initial condition, and determine $G(x)$ for $x < 0$ from the boundary condition.

4. (15 points) Consider Poisson equation with Robin boundary condition

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ \frac{\partial u}{\partial \mathbf{n}}(x) + \alpha(x) u(x) = g(x) & \text{on } \partial\Omega \end{cases}$$

where Ω is a smooth, bounded domain. α is a continuous positive functions on $\partial\Omega$. Prove that there is at most one classical solution which solves the boundary value problem.

5. (25 points) Consider the following 1D equation

$$-u'' + u = f(x), \quad -\infty < x < \infty,$$

where f is a bounded smooth function in $\mathbb{R}$.

(a) Define $\Phi(x) = \frac{1}{2} e^{-|x|}$. Verify that $-\Phi''(x) + \Phi(x) = 0$ for all $x \in \mathbb{R} \backslash \{0\}$. (Note that at $x = 0$, Φ'' is not well-defined. Φ is called the fundamental solution for $-u'' + u = 0$.)

(b) Define $u = \Phi * f$. Prove that u is a smooth solution of the equation.
Hint: To verify u satisfies the equation, one can write

$$u''(x) = \int_{-\infty}^{\infty} \Phi(y) f''(x-y) dy = \int_{-\infty}^{0} \frac{1}{2} e^{y} f''(x-y) dy + \int_{0}^{\infty} \frac{1}{2} e^{-y} f''(x-y) dy.$$

Perform integration by parts twice for each terms. The procedure is similar to but much easier than Poisson equation, not only because the equation is in 1D, but also because the fundamental solution is not singular (only not differentiable at origin).

