CAAM/MATH423 Partial Differential Equations I Fall 2015

Homework 8, Due on Tuesday, October 27, 2015

1. *(Loss of smoothness for solutions for Laplace equation at the boundary)* Finish excercise 9 in section 2.5 of Evans book. Let u be the solution of

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^n_+ \\ u = g & \text{on } \partial\mathbb{R}^n_+ \end{cases}$$

given by Poisson's formula for the half space. Assume g is bounded and $g(x) = |x|$ for $x \in \partial\mathbb{R}^n_+, |x| \leq 1$. Show Du is *not* bounded near $x = 0$. *Hint: estimate $\partial_{x_n} u(x)$ at point $x = 0$.*

2. *(Solving heat equation using Fourier transform)* Consider the initial value problem for heat equation

$$\begin{cases} u_t - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0, \infty) \\ u = g & \text{on } \mathbb{R}^n \times \{t = 0\}. \end{cases}$$

a). Take Fourier transform in of the equation in x, and solve the initial value problem for $\hat{u}$.

b). *(This part is optional, as it requires calculations in the complex plane.)* Verify that $\mathcal{F}^{-1}(e^{-|y|^2 t})(x) = \frac{1}{(2t)^{n/2}} e^{-\frac{|x|^2}{4t}}$ for $t > 0$, where the inverse Fourier transform $\mathcal{F}^{-1}$ is with respect to y.

c). Derive the solution of the initial value problem based on a) and b).

Note: one still have to verify whether the initial condition is satisfied, as the Fourier transform is not valid for $t = 0$. The procedure is the same as discussed in class. You do not have to repeat it in the homework.

3. *(Initial-boundary value problem for heat equations)*

a). Finish excercise 15 in section 2.5 of Evans book. Consider the following initial-boundary value problem for 1D heat equation

$$\begin{cases} u_t - u_{xx} = f & \text{in } \mathbb{R}_+ \times (0, \infty) \\ u = g & \text{on } \mathbb{R}_+ \times \{t = 0\} \\ u = h & \text{on } \{x = 0\} \times [0, \infty), \end{cases}$$

For simplicity, take $f \equiv 0$ and $g \equiv 0$. Set $h : [0, \infty) \to \mathbb{R}$, with $h(0) = 0$ (this is the compatibility condition as $u(0, 0) = g(0) = h(0)$.) Derive the explicit formula of the solution

$$u(x, t) = \frac{x}{\sqrt{4\pi}} \int_0^t \frac{1}{(t - s)^{3/2}} e^{-\frac{x^2}{4(t-s)}} h(s) ds.$$

Hint: Let $v(x, t) := u(x, t) - h(t)$ and extend v to $\{x < 0\}$ by odd reflection, namely $v(x, t) = -v(-x, t)$. Use integration by parts when necessary.

b). (*Optional*) What if f and g are not zero? Derive the formula for general f, g, h, with the compatibility condition $g(0) = h(0)$.

Hint: Take $v(x, t) := u(x, t) - g(x) - h(t) + g(0)$ and repeat a).