CAAM/MATH423 Partial Differential Equations I **Fall 2015**

Homework 5, Due on Tuesday, September 29, 2015

1. *(Spherical Laplacian)* Let $f(x)$ be a function in $\mathbb{R}^n$, which is radial, namely f can be expressed as $f(x) = \phi(|x|)$, for some function $\phi(r)$ defined in $\mathbb{R}_+$. Prove that

$$\Delta f(x) = \phi_{rr}(|x|) + \frac{n-1}{|x|}\phi_r(|x|).$$

2. *(Solution for 2D wave equation)* (Optional, no need to submit) Use the Hadamard's method of descent to find a solution for initial value problem of 2D wave equation, starting with Kirchhoff's formula for 3D wave equation. (You can find the proof in Page 73-74 of Evans book.)

3. *(Lebesgue spaces)*

a). Find a function f_1 which lies in $L^1(\mathbb{R})$ but not $L^\infty(\mathbb{R})$.

b). Find a function f_2 which lies in $L^\infty(\mathbb{R})$ but not $L^1(\mathbb{R})$.

c). Prove that if $f \in L^1 \cap L^\infty(\mathbb{R}^n)$, then $f \in L^p(\mathbb{R}^n)$ for all $p \in [1, \infty]$.

d). (Optional, no need to submit) Prove that L^p space is a Banach space in $\mathbb{R}^n$.

4. *(Fourier transform)* We use Fourier transform in $\mathbb{R}$ defined by $\hat{f}(y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-ixy}dx$.

a). Prove that if f is even, then $\hat{f}$ is real.

b). Let $f(x) = 1_{[-a,a]}(x)$ be the indicator function of interval $[-a, a]$. Calculate $\hat{f}$ explicitly.

c). Consider the following initial value problem of wave equation

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & \text{in } \mathbb{R} \times (0, \infty) \\ u = 0,\ u_t = h & \text{on } \mathbb{R} \times \{t = 0\}. \end{cases}$$

Write down the corresponding equation for $\hat{u}$, and solve $\hat{u}$. (The Fourier transform is taken only in space.)

d). Use the information in b) and properties of Fourier transform to solve u.

Remark: This is another way to derive d'Alembert formula, which is also useful for other linear equations. To cover the case where $g \not\equiv 0$, one can use Stoke's rule (still remember last homework?)