CAAM/MATH423 Partial Differential Equations I **Fall 2015**

Homework 12, Due on Tuesday, November 24, 2015

1. *(Barrier function)* Finish exercise 9 in section 6.6 of Evans book. Assume u is a smooth solution of $\mathcal{L}u = -\sum_{i,j=1}^n a^{ij}x_ix_j = f$ in Ω, $u = 0$ on $\partial\Omega$, where f is bounded. Fix $x^0 \in \partial\Omega$. A *barrier* at x^0 is a C^2 function w such that

$$\mathcal{L}w \geq 1 \text{ in } \Omega, \quad w(x^0) = 0, \quad w \geq 0 \text{ on } \partial\Omega.$$

a). Let $v_1(x) = u(x) + \|f\|_{L^\infty} w(x)$. Prove that $v_1(x) \geq 0$, and the minimum is attained at x^0, and hence $\frac{\partial v_1}{\partial \mathbf{n}}(x^0) \geq 0$. *Hint: Use weak maximum principle.*

b). Similarly, let $v_2(x) = u(x) - \|f\|_{L^\infty} w(x)$. Prove that $v_2(x) \leq 0$, and the minimum is attained at x^0, and hence $\frac{\partial v_2}{\partial \mathbf{n}}(x^0) \leq 0$.

c). Use a) and b) to show that

$$\left|\frac{\partial u}{\partial \mathbf{n}}(x^0)\right| \leq \|f\|_{L^\infty} \left|\frac{\partial w}{\partial \mathbf{n}}(x^0)\right|.$$

Note: the tangential derivative of u at x^0 is 0 due to zero Dirichlet boundary condition. Therefore, the full gradient $Du(x^0)$ is controlled by the normal derivative $\frac{\partial u}{\partial \mathbf{n}}(x^0)$.

2. *(Uniqueness for Neumann boundary-value problem)* Consider u, v are two smooth solution of the Neumann boundary-value problem on a smooth domain Ω

$$\begin{cases} \mathcal{L}u = f & \text{in } \Omega \\ \frac{\partial u}{\partial \mathbf{n}} = g & \text{on } \partial\Omega, \end{cases}$$

where the elliptic operator $\mathcal{L}u = \sum_{i,j=1}^n a^{ij}u_{x_ix_j} + \sum_{i=1}^n b_iu_{x_i}$, with bounded smooth coefficients a^{ij}, b^i, f and g. Prove that u and v are differed by a constant, namely $u - v \equiv C$.
Hint: Apply strong maximum principle on $u - v$. It implies either $u - v$ is a constant, or maximum is only attained at the boundary. In the latter case, apply Hopf's lemma to deduce a contradiction.

3. *(Isentropic Euler equations)* Consider 1D Euler equation with isentropic pressure

$$\begin{cases} \rho_t + (\rho u)_x = 0 \\ (\rho u)_t + (\rho u^2 + A\rho^\gamma)_x = 0 \end{cases}$$

where $\rho \geq 0, u$ represent density and velocity respectively, $A > 0$ and $\gamma > 1$ are constants.

a). Write the system in the form $U_t + F(U)_x = 0$, where $U = (\rho, m) := (\rho, \rho u)$.

b). Calculate the matrix $DF(U)$ and find its eigenvalues. The two eigenvalues one should obtain are $u \pm \sqrt{A\gamma}\rho^{\frac{\gamma-1}{2}}$. Is the system hyperbolic?