CAAM/MATH423 Partial Differential Equations I **Fall 2015**

Homework 10, Due on Tuesday, November 10, 2015

1. *(Hölder spaces)* Let $f(x) = \sqrt{|x|}$, where $x \in \mathbb{R}$. Prove that $f \in C^{0,1/2}$.
Hint: prove that $|\sqrt{|x|} - \sqrt{|y|}| \le |x-y|^{1/2}$.

2. *(Interpolating inequalities)* This set of problems is based on excercises 2, 9, 10 in section 5.10 of Evans book. Let $u \in C_c^\infty(\Omega)$, where $\Omega \in \mathbb{R}^n$. Prove the following interpolation inequalities:

a). $\|u\|_{C^{0,\gamma}(\Omega)} \le \|u\|_{C^{0,\beta}(\Omega)}^{\frac{1-\gamma}{1-\beta}} \|u\|_{C^{0,1}(\Omega)}^{\frac{\gamma-\beta}{1-\beta}}$, for $0 < \beta < \gamma \le 1$.

b). $\|Du\|_{L^2} \le C\|u\|_{L^2}^{1/2}\|D^2u\|_{L^2}^{1/2}$.

c). $\|Du\|_{L^p} \le C\|u\|_{L^p}^{1/2}\|D^2u\|_{L^p}^{1/2}$, for $2 < p < \infty$.
Hint: Write $\int_\Omega |Du|^p dx = \int_\Omega Du \cdot Du|Du|^{p-2}dx$. *Use integration by parts and Hölder's inequality.*

d). $\|Du\|_{L^{2p}} \le C\|u\|_{L^\infty}^{1/2}\|D^2u\|_{L^p}^{1/2}$, for $1 \le p < \infty$.

3. *(H^s space is closed under multiplication)* This set of problems is based on excercises 20 and 21 in section 5.10 of Evans book. Show that $H^s(\mathbb{R}^n)$ is closed under multiplication, if $s > n/2$. Namely, for all $u, v \in H^s(\mathbb{R}^n)$, the product $uv \in H^s(\mathbb{R}^n)$.

Step 1: Prove that $\|u\|_{L^\infty(\mathbb{R}^n)} \le C_1\|\hat{u}\|_{L^1(\mathbb{R}^n)}$, where C_1 depends only on n.

Step 2: Prove that $\|\hat{u}\|_{L^1(\mathbb{R}^n)} \le C_2\|(1+|y|^s)\hat{u}\|_{L^2(\mathbb{R}^n)}$, where C_2 depends on n and s.
Hint: Write $\displaystyle\int_{\mathbb{R}^n} |\hat{u}|dy = \int_{\mathbb{R}^n} (1+|y|^s)|\hat{u}| \cdot \frac{1}{1+|y|^s}dy$ *and apply Hölder's inequality.*

Step 3: Use the results above and theorem 8 in pp. 297 in Evans book to get the estimate $\|u\|_{L^\infty(\mathbb{R}^n)} \le C_3\|u\|_{H^s(\mathbb{R}^n)}$.
Note that the embedding is in $\mathbb{R}^n$. *One can check that the Sobolev number* $sob_n(s,2) > sob_n(0,\infty)$ *if and only if* $s > n/2$.

Step 4: *(Optional)* Prove that $\|uv\|_{H^s(\mathbb{R}^n)} \le \|\hat{u}\|_{L^1(\mathbb{R}^n)}\|v\|_{H^s(\mathbb{R}^n)} + \|u\|_{H^s(\mathbb{R}^n)}\|\hat{v}\|_{L^1(\mathbb{R}^n)}$.
Hint: Write the left-hand-side in Fourier, one can get

$$\begin{aligned}\|uv\|_{H^s(\mathbb{R}^n)}^2 \le& C\int (1+|y|^s)^2|\widehat{uv}(y)|^2 dy = C'\int (1+|y|^s)^2 \left|\int \hat{u}(y-z)\hat{v}(z)dz\right|^2 dy \\ \le& C'\left[\int \left(\int (1+|y|^s)^2|\hat{u}(y-z)|^2|\hat{v}(z)|^2 dy\right)^{1/2} dz\right]^2,\end{aligned}$$

where the last inequality is due to Minkovski integral inequality. Verify the following inequality $|y|^s \le C_4(|y-z|^s + z^s)$ *and plug into the estimate above.*

Note: a better inequality can be proved where $\|\hat{u}\|_{L^1}$ is replaced by $\|u\|_{L^\infty}$.

Step 5: Use the results above to prove $\|uv\|_{H^s(\mathbb{R}^n)} \le C\|u\|_{H^s(\mathbb{R}^n)}\|v\|_{H^s(\mathbb{R}^n)}$.