CAAM/MATH423 Partial Differential Equations I — Fall 2015

Final Exam — **Solution**

1. (20 points) Find the explicit local solution for the following initial value problem

$$\begin{cases} u_t + \frac{(u_x)^2+x^2}{2} = 0, & x \in \mathbb{R}, \quad t > 0, \\ u(x, t=0) = \frac{x^2}{2}. \end{cases}$$

Solution: The equation has the form $F(x,t,z,p_1,p_2) = p_2 + (p_1^2 + x^2)/2 = 0$. The system of characteristic paths starting from $(\alpha, 0)$ reads

$$\begin{cases} \dot{x} = p_1 \\ \dot{t} = 1 \\ \dot{z} = p_1^2 + p_2 \\ \dot{p}_1 = -x \\ \dot{p}_2 = 0 \end{cases} \qquad \text{subject to initial condition} \qquad \begin{cases} x(0) = \alpha \\ t(0) = 0 \\ z(0) = \frac{\alpha^2}{2} \\ p_1(0) = \alpha \\ p_2(0) = -\alpha^2 \end{cases}$$

where $p_1(0)$ is computed from initial data, and $p_2(0)$ is computed from the equation.

To solve the ODE system, we first observe $t(s) = s$. Therefore, the parameter is simply t. Also $p_2(t) = -\alpha^2$. The coupled dynamics (x, p_1) yields $x(t) = \alpha(\cos t + \sin t)$ and $p_1(t) = \alpha(\cos t - \sin t)$. Plug in everything to the z equation and get $\dot{z} = -\alpha^2 \sin(2t)$, and therefore $z(t) = \frac{\alpha^2}{2}\cos(2t)$. The solution along the characteristic path starting from $(\alpha, 0)$ reads

$$u(x(t), t) = \frac{\alpha^2}{2}\cos(2t).$$

Now, we invert the map $(\alpha, t) \to (x, t)$, and get $\alpha = \frac{x}{\cos t + \sin t}$. Therefore, the solution of the initial value problem is

$$u(x,t) = \frac{x^2}{2(\cos t + \sin t)^2}\cos(2t) = \frac{x^2\cos(2t)}{2(1+\sin(2t))}.$$

Remark: From the expression, one can see that the solution exists for $t < 3\pi/4$.

In the distributed problem set, the initial condition is mistakenly written as $g(x)$. In this case, the solution does not have a clean expression. One gets

$$\begin{cases} x(t) = \alpha\cos t + g'(\alpha)\sin t \\ p_1(t) = -\alpha\sin t + g'(\alpha)\cos t \\ p_2(t) = -\frac{1}{2}(g'(\alpha)^2 + \alpha^2) \\ z(t) = (g'(\alpha)^2 - \alpha^2)\frac{\sin(2t)}{2} + \alpha g'(\alpha)\frac{\cos(2t)-1}{2} + g(\alpha) \end{cases}$$

Giving (x,t), let $F(\alpha) = \alpha\cos t + g'(\alpha)\sin t$. Then,

$$u(x,t) = (g'(F^{-1}(x))^2 - F^{-1}(x)^2)\frac{\sin(2t)}{2} + F^{-1}(x)g'(F^{-1}(x))\frac{\cos(2t)-1}{2} + g(F^{-1}(x)).$$

Note that $F'(\alpha) = \cos t + g''(\alpha)\sin t$. At $t = 0$, $F'(\alpha) = 1$ for all α. Therefore, $F^{-1}(x)$ is well-defined for small t, and $u(x,t)$ defined above is a valid local solution.

2. (15 points) Let $h \in C_0^\infty(\mathbb{R}^n)$, and let u be the solution to

$$\begin{cases} u_{tt} - \Delta u = 0, & x \in \mathbb{R}^n,\ t > 0, \\ u(x, t = 0) = 0, & u_t(x, t = 0) = h(x). \end{cases}$$

Let $v(x, t) = \int_0^t u(x, s) ds$. Prove

$$\int_{\mathbb{R}^n} (\Delta v(x, t))^2 dx \le 4\|h\|_{L^2}^2.$$

Hint: Use Fourier transform in x. First solve $\hat{u}(y, t)$ and $\hat{v}(y, t)$. Then use Plancherel's theorem.

Solution: Take Fourier transform for the wave equation, we get

$$\begin{cases} \hat{u}_{tt} + |y|^2 u = 0, & y \in \mathbb{R}^n,\ t > 0, \\ \hat{u}(y, t = 0) = 0, & \hat{u}_t(y, t = 0) = \hat{h}(y). \end{cases}$$

We can solve the system for each y and obtain $\hat{u}(y, t) = \hat{h}(y)\frac{\sin(|y|t)}{|y|}$ and therefore

$$\hat{v}(y, t) = \int_0^t \hat{h}(y)\frac{\sin(|y|s)}{|y|} ds = -\left.\frac{\hat{h}(y)\cos(|y|s)}{|y|^2}\right|_0^t = \frac{\hat{h}(y)}{|y|^2}(1 - \cos(|y|t)).$$

Use Plancherel's theorem, we get

$$\int_{\mathbb{R}^n} (\Delta v(x, t))^2 dx = \int_{\mathbb{R}^n} \left(-|y|^2 \hat{v}(y, t)\right)^2 dy = \int_{\mathbb{R}^n} |\hat{h}(y)|^2 (1 - \cos(|y|t))^2 dy \le 4 \int_{\mathbb{R}^n} |\hat{h}(y)|^2 dy = 4\|h\|_{L^2}^2.$$

3. (15 points) Prove that there exists a constant C, depending only on dimension n, such that

$$\max_{B(0,1)} |u| \le C \left(\max_{\partial B(0,1)} |g| + \max_{B(0,1)} |f| \right)$$

whenever u is a smooth solution of

$$\begin{cases} \Delta u = f & \text{in } B(0, 1) \\ u = g & \text{on } \partial B(0, 1). \end{cases}$$

Hint: Let $M = \max_{B(0,1)} |f|$. Define $v(x) = u(x) + \frac{M}{n}|x|^2$. Prove that v is subharmonic. Then apply maximum principle to v.

Solution: Define $v(x) = u(x) + \frac{M}{2n}|x|^2$. Compute $-\Delta v = -f - M \le 0$. So v is subharmonic. By strong maximum principle, we get

$$\max_{x \in B(0,1)} u(x) \le \max_{x \in B(0,1)} v(x) = \max_{x \in \partial B(0,1)} v(x) = \max_{\partial B(0,1)} g(x) + \frac{M}{2n}.$$

On the other hand, define $w(x) = -u(x) + \frac{M}{2n}|x|^2$. Compute $-\Delta w = f - M \le 0$. So w is also subharmonic. By strong maximum principle, we get

$$\max_{x\in B(0,1)} -u(x) \le \max_{x\in B(0,1)} w(x) = \max_{x\in\partial B(0,1)} w(x) = \max_{\partial B(0,1)} (-g(x)) + \frac{M}{2n}.$$

Put the two estimates together, we conclude

$$\max_{x\in B(0,1)} |u(x)| \le \max_{\partial B(0,1)} |g(x)| + \frac{1}{2n} \max_{B(0,1)} |f|.$$

The inequality is proved with $C = 1$.

4. (10 points) Consider heat equation with a source term and a Dirichlet boundary condition

$$\begin{cases} u_t - \Delta u = f & \text{in } \Omega \times (0, +\infty) \\ u(x,0) = g(x) & \text{for } x \in \Omega,\ t = 0 \\ u(x,t) = h(x,t) & \text{for } x \in \partial\Omega,\ t \in (0,\infty) \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^n$. Prove that there is at most one classical solution which solves the initial-boundary value problem.

Solution: Let u, v be two classical solutions of the initial-boundary value problem. Take $w = u - v$. Then w satisfies

$$\begin{cases} w_t - \Delta w = 0 & \text{in } \Omega \times (0, +\infty) \\ w(x,0) = 0 & \text{for } x \in \Omega,\ t = 0 \\ w(x,t) = 0 & \text{for } x \in \partial\Omega,\ t \in (0,\infty) \end{cases}$$

Multiply the heat equation by w and integrate in Ω, we obtain the following energy estimate

$$E(t) := \frac{1}{2}\frac{d}{dt}\int_\Omega w(x,t)^2 dx = \int_\Omega w(x,t)\Delta w(x,t) dx = -\int_\Omega |Dw|^2 dx \le 0,$$

where we use the zero Dirichlet boundary condition when performing integration by parts. This implies the energy is not increasing, i.e. $E(t) \le E(0)$. From initial condition, we know $E(0) = 0$. Therefore, since $E(t)$ is non-negative, we conclude $E(t) = 0$, namely $w = 0$ almost everywhere. Since u, v are classical solutions, $w - u - v$ is continuous. Therefore, $w \equiv 0$ and $u \equiv v$. This implies uniqueness.

5. (10 points) Let Ω be a bounded open set in $\mathbb{R}^3$. Suppose $u \in H^1(\Omega)$. Prove that $u^3 \in L^2(\Omega)$.

Solution: By Sobolev embedding $H^1(\Omega) \subset L^6(\Omega)$, as the Sobolev numbers $sob_3(1,2) = sob_3(0,6) = -1/2$ matches. Therefore, $u \in L^6(\Omega)$, namely $\int_\Omega |u|^6 dx < \infty$. Equivalently $\int_\Omega |u^3|^2 dx < \infty$, which means $u^3 \in L^2(\Omega)$.

6. (20 points) Let Ω be a smooth bounded open domain in $\mathbb{R}^2$, and f is a continuous function in $\bar{\Omega}$. Find the weak formulation of the equation

$$-\Delta u + \frac{u}{3} + u_x = f$$

where $(x, y) \in \Omega$ with zero Dirichlet boundary condition. Show that the weak problem has a unique solution.

Solution: The weak formulation of the equation states

$$\int_\Omega \left(Du \cdot Dv + \frac{1}{3} uv + u_x v \right) dxdy = \int_\Omega fvdxdy, \quad \forall\ v \in H_0^1(\Omega).$$

We apply Lax-Milgram theorem to prove existence and uniqueness of weak solution. To this end, we verify the assumptions in the theorem.

Let $B(u,v) = \int_\Omega \left(Du \cdot Dv + \frac{1}{3}uv + u_x v \right) dxdy$. We first prove B is a bounded bilinear operator in $H_0^1(\Omega)$:

$$|B(u,v)| \leq \|Du\|_{L^2}\|Dv\|_{L^2} + \frac{1}{3}\|u\|_{L^2}\|v\|_{L^2} + \|u_x\|_{L^2}\|v\|_{L^2} \leq \frac{7}{3}\|u\|_{H^1}\|v\|_{H^1}.$$

Here we use Hölder inequality.

Next, we prove B is coercive. From Cauchy-Schwarz inequality, we get

$$\int_\Omega |u_x u| dxdy \leq \int_\Omega \left(\frac{1}{2c}|u_x|^2 + \frac{c}{2}|u|^2 \right) dxdy \leq \frac{1}{2c}\|Du\|_{L^2}^2 + \frac{c}{2}|u|_{L^2}^2,$$

The inequality is true for any $c > 0$. Using this inequality, we obtain

$$B(u,u) = \int_\Omega \left(|Du|^2 + \frac{u^2}{3} + u_x u \right) dxdy \geq \left(1 - \frac{1}{2c}\right)\|Du\|_{L^2}^2 + \left(\frac{1}{3} - \frac{c}{2}\right)\|u\|_{L^2}^2 \geq \beta\|u\|_{H^1}^2,$$

where $\beta = \min(1 - \frac{1}{2c}, \frac{1}{3} - \frac{c}{2})$. Take $c \in (\frac{1}{2}, \frac{2}{3})$, then $\beta > 0$. (For example, take $c = .6$, then $\beta = .1$)

Finally, f is continuous, then clearly $\int_\Omega fvdxdy$ is bounded.

Lax-Milgram theorem then implies existence and uniqueness of weak solution.

7. (20 points) Find the explicit entropy solution of the following equation

$$u_t + \left(\frac{u^2}{2} + u \right)_x = 0, \quad x \in \mathbb{R},\ t \geq 0,$$

with initial condition

$$u(x, t = 0) = g(x) = \begin{cases} -2 & -1 < x < 0 \\ 0 & \text{otherwise} \end{cases}.$$

Solution: The flux $F(u) = \frac{u^2}{2} + u$, and the wave speed $F'(u) = u + 1$. Initially, we have

$$F'(u(x,0)) = \begin{cases} -1 & -1 < x < 0 \\ 1 & \text{otherwise} \end{cases}.$$

Therefore, there is a shock wave at $x = -1$, and a rarefaction wave at $x = 0$.

For the shock wave, the speed of the shock satisfies the RankineHugoniot condition

$$\begin{cases} \dot{\sigma} = \frac{F(-2)-F(0)}{-2-0} = 0 \\ \sigma(0) = -1 \end{cases} \quad \Rightarrow \quad \sigma(t) = -1.$$

For the rarefaction wave, the fan is $\frac{x}{t} \in [-1, 1]$. Inside the fan, $u(x,t) = (F')^{-1}(\frac{x}{t}) = \frac{x}{t} - 1$. Note that $(F')^{-1}(y) = y - 1$.

The rarefaction fan touches the shock discontinuity at $t = 1$. So before they meet, the entropy solution reads

$$u(x,t) = \begin{cases} 0 & x < -1 \\ -2 & -1 < x < -t \\ \frac{x}{t} - 1 & -t < x < t \\ 0 & x > t \end{cases}, \quad t \le 1.$$

After $t = 1$, the shock interacts with the rarefaction fan. As $F'(u_-) = 1$ and $F'(u_+) \in (-1, 1)$, the entropy solution admits a shock. The RankineHugoniot condition says

$$\begin{cases} \dot{\sigma} = \frac{F(\frac{\sigma}{t}-1)-F(0)}{\frac{\sigma}{t}-1-0} = \frac{\sigma}{2t} + \frac{1}{2} \\ \sigma(1) = -1 \end{cases} \quad \Rightarrow \quad \sigma(t) = t - 2\sqrt{t}.$$

Therefore, the solution after time 1 reads

$$u(x,t) = \begin{cases} 0 & x < t - 2\sqrt{t} \\ \frac{x}{t} - 1 & t - 2\sqrt{t} < x < t \\ 0 & x > t \end{cases}, \quad t \ge 1.$$