MATH322 Introduction to Mathematical Analysis II **Spring 2016**

Midterm Exam Name: ____________________________

Instructions: *Please read carefully!!*

- Show all work clearly and in order.
- Use no more than **three** consecutive hours of your choice to finish questions 1-4.
- Use as much time as you want for question 5.
- You are allowed to use any resources, but communication with others are not allowed.
- You are required to sign the *honor pledge*.
- Please hand in the exam on March 16 in class.
- Please attach this page with your name and signature on submission.

Honor Pledge: The Rice University Honor Pledge reads:

> "On my honor, I have neither given nor received any unauthorized aid on this exam."

Please write the exact wording of the Pledge, following by your signature, in the space below:

Pledge: ____

____

Your Signature: ____

Gradebox: For grading use. Please leave it blank.

Question:	1	2	3	4	5	Total
Points:	20	20	20	20	20	100
Score:						

Good Luck

1. (20 points) Let $f \in \mathcal{C}^\infty([a,b])$. Fix an $\epsilon > 0$, and set $\Gamma(f,\epsilon) = \{x \in [a,b] : |f(x)| < \epsilon\}$. Assume $\Gamma(f,\epsilon)$ has the form

$$\Gamma(f,\epsilon) = \bigcup_{i=1}^{n} (a_i, b_i),$$

where $b_i \leq a_i$ for all $j = 1, \cdots, n-1$. Prove that there exists a point $\xi \in (a,b)$ such that $f^{(n-1)}(\xi) = 0$.

2. (20 points) Let $f : [a,b] \to \mathbb{R}$ be a bounded function which is discontinuous at a point $c \in (a,b)$. μ is a monotonically increasing function which is discontinuous at c as well. Is is true that f is not μ-integrable?

   If you answer true, namely for all f and μ that shares discontinuity at some point, $f \notin \mathcal{R}(\mu)$, prove it.

   If you answer false, construct a pair of f and μ that shares a discontinuity, and f is μ-integrable.

3. (20 points) Let $\{f_n\}$ be a sequence of $\mathcal{C}^\infty$ functions on a compact interval, such that for any integer $k \geq 0$, there exists an M_k such that $|f_n^{(k)}| \leq M_k$ for all x and n. Prove that there exists a subsequence which converges to a $\mathcal{C}^\infty$ function, in $\|\cdot\|_{\mathcal{C}^\infty}$ norm.

   Hint: apply Arzela-Ascoli theorem on $\{f_n^{(k)}\}$ and use diagonal argument. You can prove uniform convergences to get partial credit.

   Remark: The $\mathcal{C}^\infty$ norm is defined as

$$\|f\|_{\mathcal{C}^\infty([a,b])} = \sup_{k \geq 0} \max_{x \in [a,b]} |f^{(k)}(x)|.$$

4. (20 points) Let μ be a measure with a cumulative distribution function (also denoted by μ) $\mu : [a,b] \to \mathbb{R}$ which is bounded and monotonically increasing. Construct a sequence of simple functions ν_n such that the corresponding measure ν_n converges to μ in weak-$*$ topology.

   Remarks: 1. Simple function means step functions taking only finite many values. As ν_n is a cumulative distribution of a measure, it should also be an increasing function. The corresponding measure is a collection of point masses.
   2. ν_n converges to μ in weak-$*$ topology means for all $f \in \mathcal{C}([a,b])$,

$$\lim_{n\to\infty} \int_a^b f d\nu_n = \int_a^b f d\mu.$$

5. (20 points) Ask yourself a question related to the material covered in the first half of the semester. Explain why it is interesting and nontrivial. Then try to answer it.