MATH322 Introduction to Mathematical Analysis II **Spring 2016**

Homework 9, Due on Wednesday, March 30, 2016

1. *(Fourier coefficients)* Compute the Fourier coefficients explicitly for the following two 2π-periodic functions.

a). $f(x) = x$ for $x \in (-\pi, \pi]$. (The function has discontinuity at $x = \pi$.)

b). $g(x) = x^2$ for $x \in [-\pi, \pi]$. (The function has a *kink* at $x = \pi$.)

What is the decay rate of the Fourier coefficients? Which one decays faster?

c). (Optional) Find a function $h(x)$ where the decay rate of the Fourier coefficients $\hat{h}(n)$ is different from $\hat{f}(n)$ and $\hat{g}(n)$.

2. *(Embedding for H^s space)* f is a 2π-periodic function defined in $\mathbb{R}$.

a). Prove that if $\sum_{n=-\infty}^{\infty} |\hat{f}(n)|$ converges, then the Fourier series $f(x) = \sum_{n=-\infty}^{\infty} \hat{f}(n)e^{-inx}$ converges pointwise, and moreover,

$$\|f\|_{L^\infty} \leq \sum_{n=-\infty}^{\infty} |\hat{f}(n)|.$$

b). Prove that for $f \in H^s$,

$$\sum_{n=-\infty}^{\infty} |\hat{f}(n)| \leq C\|f\|_{H^s},$$

where the constant C only depend on s and it is finite when $s > 1/2$.

Hint: Write $\|f\|_{H^s}^2 = \sum_{n=-\infty}^{\infty}(1+n^2)^s|\hat{f}(n)|^2$, and use Cauchy-Schwarz inequality.

c). Use a) and b) to conclude that H^s is a subspace of L^∞.

Remark: For any $f \in H^s$, we know $f \in L^2$. This does not imply $f \in L^\infty$ as L^∞ is a smaller space than L^2. The embedding theorem above says that if f has some extra differentiability, then f lies in a smaller Lebesgue space. It connects differentiability and integrability.

3. *(Pointwise divergence of Fourier series)* (Hard and optional) Find a continuous periodic function f and a point x such that

$$\limsup_{N\to\infty} |s_N(x; f)| = \infty.$$