MATH322 Introduction to Mathematical Analysis II **Spring 2016**

Homework 8, Due on Wednesday, March 23, 2016

1. *(Convolution)* Let f, g be continuous 2π-periodic functions. The convolution of these two functions, denoted by $f * g$, is defined as

$$f * g(x) = \int_{-\pi}^{\pi} f(x-y)g(y)dy.$$

a). prove that $f * g$ is 2π-periodic.

b). Use change of variable formula and periodicity to prove $f * g(x) = \int_{-\pi}^{\pi} f(y)g(x-y)dy$.

c). Let $\hat{f}_n, \hat{g}_n$ be the n-th Fourier coefficient of f and g respectively. Prove that the n-th Fourier coefficient of $f * g$ is $2\pi \hat{f}_n \hat{g}_n$.

2. *(Approximation by trigonometric polynomial)* The Fejér kernel is defined as

$$F_N(x) = \frac{1}{N+1} \sum_{n=0}^{N} D_n(x),$$

where D_n represents Dirichlet kernel.

a). Prove that

$$F_N(x) = \frac{1}{N+1} \cdot \frac{1-\cos(N+1)x}{1-\cos x}$$

and following properties holds:

$$F_N \geq 0, \quad \frac{1}{2\pi}\int_{-\pi}^{\pi} F_N(x)dx = 1, \quad \text{and } F_N(x) \leq \frac{1}{N+1} \cdot \frac{2}{1-\cos\delta}, \text{ if } 0 < \delta \leq |x| \leq \pi.$$

b). For $x \in [-\pi, \pi]$, let

$$\mu(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}, \quad \nu_n(x) = \frac{1}{2\pi}\int_{-\pi}^{x} F_N(y)dy.$$

Prove that $d\nu_n$ converges to $d\mu$ in weak-$*$ topology.

Hint: take the same procedure as problem 1 in homework 7. Use the properties in a).

c). Given f is a continuous 2π-periodic function. Define $T_n(x) = \int_{-\pi}^{\pi} f(x-y)d\nu_n(y)$. Prove that T_n is a trigonometric polynomial, and T_n converges to f uniformly.

Hint: Write $f(x)$ as $\int_{-\pi}^{\pi} f(x-y)d\mu(y)$, and apply b).

d). (Optional) If f is not continuous and has a jump discontinuity at x. Prove that

$$\lim_{n\to\infty} T_n(x) = \frac{1}{2}\big(f(x+) + f(x-)\big).$$