MATH322 Introduction to Mathematical Analysis II **Spring 2016**

Homework 7, Due on Wednesday, March 9, 2016

1. *(Convergence in weak-$*$ topology)* Let $\nu_n, \mu : [a,b] \to \mathbb{R}$ be monotonically increasing functions. We say that $d\nu_n$ converges to $d\mu$ in weak-$*$ topology as n goes to infinity if the following holds: for each $f \in \mathcal{C}([a,b])$,

$$\lim_{n\to\infty} \int_a^b f d\nu_n = \int_a^b f d\mu.$$

Let $[a,b] = [-1,1]$,

$$\mu(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}, \quad \nu_n(x) = \int_{-1}^x c_n(1-x^2)^n dx, \quad \text{where } c_n = \left[\int_{-1}^1 (1-x^2)^n dx\right]^{-1}.$$

Prove that $d\nu_n$ converges to $d\mu$ in weak-$*$ topology. And use this to prove Stone-Weirstrass theorem.

2. *(An application of Stone-Weirstrass theorem)* Finish exercise 20 in chapter 7 of Rudin's book. If f is continuous on $[0,1]$ and if

$$\int_0^1 f(x)x^n dx = 0, \quad \forall\ n = 0,1,2,\cdots,$$

prove that $f(x) = 0$ on $[0,1]$. *Hint: The integral of the product of f with any polynomial is zero. Use the Weirstrass theorem to show that $\int_0^1 f^2(x)dx = 0$.*

3. *(Convergence with information of the derivatives)* Let $\{f_n\}$ be a uniformly bounded sequence of real-valued differentialable functions on $[a,b]$ such that the derivatives $\{f_n'\}$ are uniformly bounded as well. Prove that there exists a subsequence $\{f_{n_k}\}$ that converges uniformly on $[a,b]$. *Hint: Use Arzela-Ascoli theorem.*