MATH322 Introduction to Mathematical Analysis II **Spring 2016**

Homework 5, Due on Wednesday, February 17, 2016

1. *(Improper integrals)* Finish exercises 7 and 8 in chapter 6 of Rudin's book. We can extend the definition of Riemann integrable function to unbounded functions and infinite domains.

2. *(L^p space)* In this problem, we introduce a set of functional spaces. The space $L^p([a,b])$ ($p \geq 1$, L stands for "Lebesgue") is a collection of functions f such that

$$\int_a^b |f(x)|^p dx$$

exists and is finite. The space L^p is equipped with norm

$$\|f\|_{L^p([a,b])} = \left(\int_a^b |f(x)|^p dx\right)^{1/p}.$$

a). If we only consider proper integrals, namely $[a,b]$ is a compact interval, and f is bounded on $[a,b]$. Then, all L^p spaces are equivalent. *Hint: you need to prove that if $|f|^p$ is Riemann integrable, then $|f|^q$ is Riemann integrable as well, for all $p,q \geq 1$.*

b). If we consider L^p space in infinite interval $[a,\infty)$, then L^p spaces are not equivalent. Find a bounded function f such that $f \in L^2([0,\infty))$ but $f \notin L^1([0,\infty))$.

c). Prove that if f is bounded, then $L^p([a,\infty))$ has an order. Namely, if $f \in L^p([a,\infty))$, then $f \in L^q([a,\infty))$ for all $q > p$.

d). Prove the following *Hölder's inequality.*

$$\left|\int_a^b g(x)h(x)dx\right| \leq \|g\|_{L^r([a,b])}\|h\|_{L^{r^*}([a,b])}, \quad \text{for } \frac{1}{r} + \frac{1}{r^*} = 1.$$

Hint: One can go through exercise 10 (a)-(d) in chapter 6 of Rudin's book to prove the inequality. Note that the inequality stays true for improper integrals as well.

e). If we allow functions to be unbounded, then L^p spaces are not equivalent. Find an example that $f \in L^1([0,1])$ but $f \notin L^2([0,1])$.

f). (Bonus) Prove that if the interval $[a,b]$ is compact, then $L^p([a,b])$ has an order. More precisely, if $f \in L^p([a,b])$, then $f \in L^q([a,b])$ for all $1 \leq q < p$. *Hint: apply Hölder's inequality for $g(x) = |f(x)|^q$ and $h(x) = 1$. The left hand side becomes $\int_a^b |f(x)|^q dx$. Find an appropriate pair of power (r, r^*) so that the right hand side is finite.*

Note that we have different orderings of L^p spaces for the case if f is bounded and the case $[a,b]$ is compact. In the former case, L^1 is smaller than L^2, but in the later case, L^1 is larger than L^2. If we allow both f to be unbounded and the domain to be non-compact, then L^p spaces have no ordering.
The L^p spaces reflect the integrability of a function, while $\mathcal{C}^k$ spaces (as well as Hölder spaces) reflect the differentiability of a function. They are different spaces. It is a very interesting topic to understand whether there is any ordering between these two types of spaces.