MATH322 Introduction to Mathematical Analysis II **Spring 2016**

Homework 4, Due on Wednesday, February 10, 2016

1. *(Uniqueness of Riemann integral)* Show that if the Riemann integral of f exists, then it is unique. *Hint: prove by contradiction. The ϵ-P definition might be useful.*

2. *(Calculate Riemann integral)* Prove that the function $f(x) = x^2$ is Riemann integrable on [0,1] and show that

$$\int_0^1 x^3 dx = \frac{1}{4}.$$

Hint: try to prove that for all $\epsilon > 0$, there exist a partition P such that $0 < U(f,P) - \frac{1}{4} < \epsilon$ and $0 < \frac{1}{4} - L(f,P) < \epsilon$. You can pick the partition to be equally spaced, and use the identity $\sum_{i=1}^{n} i^3 = \frac{1}{4}(n^4 + 2n^3 + n^2)$.

3. *(Integrability for discontinuous functions)* We know from class that if f is bounded and contains finite many discontinuities, then f is Riemann integrable. This problem discuss the integrability of functions with infinite many discontinuities.

a). We call an infinite set $\{x_i\} \subset [a,b]$ has *measure zero* if for any $\epsilon > 0$, there exists an open set V, such that $x_i \in V$ for all i, and $|V| < \epsilon$. ($|V|$ denotes the size of V.) Prove that if $\{x_i\}$ has finite many limit points, then $\{x_i\}$ has measure zero.

b). Prove that if f is bounded and has a set of discontinuities of measure zero on $[a,b]$, then f is Riemann integrable. *Hint: find a partition P where the "bad part" is chosen as V in part a).*

From part a) and b), we know that if f has infinite many discontinuities with finite many limiting points, then f is Riemann integrable. The next question is whether the assumption on finite many limiting points can be removed.

c). Find and verify a Riemann integrable function f with (uncountable) infinite many discontinuities, and the set of discontinuities has infinite many limits. *Hint: There exists an infinite set which is perfect and has measure zero.*

So, the punchline for integrability is whether the set of discontinuities is measure zero or not. Indeed, for the typical bounded function which is not Riemann integrable $f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$ in $[0,1]$, the set of discontinuities is $[0,1]$, which does not have measure zero. Let's end up the discussion with a fun question.

d). Let f be a function defined in $[0,1]$ by

$$f(x) = \begin{cases} 1/q & \text{if } x = p/q \text{ is rational in lowest terms and } q > 0 \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

(in lowest term means $f(4/6) = 1/3$ as $4/6 = 2/3$.) Is f integrable? If yes, prove it and what is $\int_0^1 f(x)dx$? If no, disprove it (calculate $\underline{\int_0^1} f(x)dx$ and $\overline{\int_0^1} f(x)dx$ and verify they do not match).