MATH322 Introduction to Mathematical Analysis II **Spring 2016**

Homework 2, Due on Wednesday, January 27, 2016

1. *(Hölder space)* A Hölder space $\mathcal{C}^{0,\alpha}([a,b])$ for $\alpha \in (0,1)$ is defined as the set of all functions f such that there exists a finite number C such that for all $x, y \in [a,b]$,

$$|f(x) - f(y)| \leq C|x-y|^\alpha.$$

a). Prove that $\mathcal{C}^{0,\alpha}([a,b])$ is a proper subspace of $\mathcal{C}([a,b])$, namely any function in Hölder space is continuous, bur not all continuous functions are Hölder. *Hint: To prove $\mathcal{C}^{0,\alpha} \subset \mathcal{C}$, use ϵ-δ language to make it clear. You can mimic the proof that differentiable functions are continuous. To prove $\mathcal{C}^{0,\alpha} \neq \mathcal{C}$, consider the conter example $f(x) = |x|^{\alpha/2}$, $x \in [-1,1]$. Check f is continuous but not Hölder around zero.*

b). (Optional) Prove that $\mathcal{C}^1([a,b])$ is a proper subspace of $\mathcal{C}^{0,\alpha}([a,b])$. *Note: You do not have to submit this part as the proof should be quite similar to part a). To find a function which is $\mathcal{C}^{0,\alpha}$ but not $\mathcal{C}^1$, try $f(x) = |x|^{(\alpha+1)/2}$, $x \in [-1,1]$.*

Hölder spaces extend the family of differentiable spaces $\mathcal{C}^k$ to non-integer powers, which turns out to be very powerful to measure the differentiability of a function. Several important remarks are as follows.

c). (Optional) $\mathcal{C}^{0,\alpha}$ is a proper subspace of $\mathcal{C}^{0,\beta}$ if $\alpha > \beta$.

d). The space $\mathcal{C}^{0,1}$ is called *space of Lipschitz continuous functions.* Prove that $\mathcal{C}^1([a,b])$ is a proper subspace of $\mathcal{C}^{0,1}([a,b])$. *Hint: Use mean value theorem to prove $\mathcal{C}^1 \subset \mathcal{C}^{0,1}$. For conter example, consider $f(x) = |x|$, $x \in [-1,1]$.*

e). (Bonus point) Prove that for $\alpha > 1$, the space $\mathcal{C}^{0,\alpha}$ only contains constant functions. *Hint: First, prove that if $f \in \mathcal{C}^{0,\alpha}$ for $\alpha > 1$, then $f \in \mathcal{C}^1$. Then use mean value theorem to prove $f' \equiv 0$.*

Therefore, to link $\mathcal{C}^1$ and $\mathcal{C}^2$, one should not use $\mathcal{C}^{0,\alpha}$, but rather use $\mathcal{C}^{1,\alpha}$ with $\alpha \in (0,1)$, namely $|f'(x) - f'(y)| \leq C|x-y|^\alpha$. The spaces $\mathcal{C}^{k,\alpha}$ are all called Hölder spaces.

2. *(3 point scheme on second derivative)* Finish exercise 11 in chapter 5 of Rudin's book. Suppose f is defined in a neighborhood of x, and suppose $f''(x)$ exists. Use l'Hôpital's rule to show that

$$\lim_{h\to 0} \frac{f(x+h) + f(x-h) - 2f(x)}{h^2} = f''(x).$$

Show by an example that the limit may exist even $f''(x)$ does not.

3. *(Fixed point theorem)* Finish exercise 22 (a)-(c) in chapter 5 of Rundin's book.
Hint for (c): Write $x_{n+1} - x_n = f(x_n) - f(x_{n-1})$. Apply mean value theorem and then prove the sequence $\{x_n\}$ is Cauchy.