MATH322 Introduction to Mathematical Analysis II **Spring 2016**

Homework 11, Due on Wednesday, April 20, 2016

1. *(Chain rule on partial derivatives)* Let $f : \mathbb{R}^n \to \mathbb{R}^m$ and $g : \mathbb{R}^m \to \mathbb{R}^l$ be $\mathcal{C}^1$ mappings. The composition $g \circ f$ is defined as a mapping from $\mathbb{R}^n$ to $\mathbb{R}^l$ such that $g \circ f(x) = g(f(x))$. Prove that $g \circ f$ is a $\mathcal{C}^1$ mapping, and moreover, for any $x \in \mathbb{R}^n$, $k = 1, \cdots, n$, $i = 1, \cdots, l$,

$$\frac{\partial (g \circ f)^i}{\partial x_k}(x) = \sum_{j=1}^{m} \frac{\partial g^i}{\partial y_j}(f(x)) \cdot \frac{\partial f^j}{\partial x_k}(x).$$

2. *(Invertibility of a linear transformation)* Let A be a linear transformation from vector space $X \in \mathbb{R}^n$ to $\mathbb{R}^m$, where $dim(X) = r$.

- **a)**. Prove that the range of A, defined as $\mathcal{R}(A) = \{A(x) : x \in X\}$, is a vector space, with $dim(\mathcal{R}(A)) \leq r$.
- **b)**. Prove that A is invertible if and only if $dim(\mathcal{R}(A)) = r$. In particular, this implies that A can not be invertible if $m < r$.
- **c)**. (Optional, but not hard) Show that A can be uniquely determined by an r-by-r matrix. Express how to find $A(x)$ given the matrix.

3. *(Local invertibility versus global invertibility)* Suppose $f : \mathbb{R}^n \to \mathbb{R}^m$ is a $\mathcal{C}^1$ mapping, f is locally invertible for all $x \in \mathbb{R}^n$.

- **a)**. Is f invertible in $\mathbb{R}^n$? If yes, prove it. If no, find a counter example.
- **b)**. What if we restrict $n = 1$? If yes, prove it. If no, find a counter example.
- **c)**. What if we restrict $m = 1$? If yes, prove it. If no, find a counter example.