MATH322 Introduction to Mathematical Analysis II **Spring 2016**

Final Exam **Solution**

1. (20 points) Let f be a positive function defined on $[0,+\infty)$, such that $f \in \mathcal{R}$ on $[0,A]$ for all $A < +\infty$. Moreover, $\lim_{x\to+\infty} f(x) = 1$. The Laplace transform of f, denoted by $\mathcal{L}[f]$, is a function defined on $[0,+\infty)$, with

$$\mathcal{L}[f](\lambda) = \int_0^\infty e^{-\lambda x} f(x)dx.$$

(a) Prove that $\mathcal{L}[f](\lambda)$ is well-defined for all $\lambda > 0$. Namely, the improper integral converges.

Solution: Since $\lim_{x\to+\infty} f(x) = 1$, we get $\exists x_0$ such that $f(x) \in (\frac{1}{2}, 2)$ for all $x > x_0$. Therefore, for $\lambda > 0$,

$$\mathcal{L}[f](\lambda) = \int_0^{x_0} e^{-\lambda x} f(x)dx + \int_{x_0}^\infty e^{-\lambda x} f(x)dx < \int_0^{x_0} f(x)dx + \int_{x_0}^\infty 2e^{-\lambda x}dx.$$

The first integral is finite as f is integrable on $[0, x_0]$. The second one is also finite as

$$\int_{x_0}^\infty 2e^{-\lambda x}dx = \frac{2}{\lambda}e^{-\lambda x_0}.$$

Therefore, the improper integral converges and $\mathcal{L}[f](\lambda)$ is well-defined.

(b) Prove that $\mathcal{L}[f](0)$ diverges. Moreover,

$$\lim_{\lambda\to 0} \lambda\mathcal{L}[f](\lambda) = 1.$$

Solution:

$$\mathcal{L}[f](0) = \int_0^{x_0} f(x)dx + \int_{x_0}^\infty f(x)dx \geq 0 + \int_{x_0}^\infty \frac{1}{2}dx,$$

where x_0 is the same as part (a). Clearly, it diverges.

For the next limit, we separate the integral at a different point. Given $\epsilon > 0$, the goal is to prove that $\exists\delta > 0$, for all $\lambda < \delta$, $|\lambda\mathcal{L}[f](\lambda) - 1| < \epsilon$.

First, pick x_1 such that $f(x) \in (1 - \epsilon/3, 1 + \epsilon/3)$ for all $x > x_1$. Next, compute

$$\lambda\mathcal{L}[f](\lambda) - 1 = \lambda \int_0^{x_1} e^{-\lambda x} f(x)dx + \left(\lambda \int_{x_1}^\infty e^{-\lambda x} f(x)dx - 1\right) = I + II.$$

For the first term,

$$|I| \leq \lambda \int_0^{x_1} f(x)dx < \frac{\epsilon}{3},$$

if we pick δ small enough, more precisely $\delta < (3\int_0^{x_1} f(x)dx)^{-1}\epsilon$.

For the second term,

$$
\begin{aligned}
|II| = & \left|\lambda \int_{x_1}^{\infty} e^{-\lambda x}(f(x)-1)dx + \left(\lambda \int_{x_1}^{\infty} e^{-\lambda x}dx - 1\right)\right| \\
\leq & \frac{\epsilon}{3} e^{-\lambda x_1} + |e^{-\lambda x_1} - 1| \leq \frac{\epsilon}{3} + \frac{\epsilon}{3},
\end{aligned}
$$

where in the last inequality, we take $\delta < -x_1^{-1}\log(1-\frac{\epsilon}{3})$.
Finally, we conclude that if $\lambda < \delta$, $|\lambda\mathcal{L}[f](\lambda) - 1| \leq |I| + |II| < \epsilon$.

2. (20 points) Let f be a 2π-periodic function with $f(x) = (\pi - |x|)^2$ on $[-\pi, \pi]$.

(a) Find the Fourier coefficients $\hat{f}(n)$ for $n \in \mathbb{Z}$ and write f in terms of Fourier series.

Solution:

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx = \frac{1}{2\pi}\int_0^{\pi}(\pi - x)^2(e^{-inx} + e^{inx})dx = \frac{1}{\pi}\int_0^{\pi}(\pi - x)^2\cos(nx)dx.$$

For $n = 0$, we get $\hat{f}(0) = \frac{\pi^2}{3}$. For $n \neq 0$, do integration by parts and get $\hat{f}(n) = \frac{2}{n^2}$. Therefore,

$$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{2}{n^2}(e^{inx} + e^{-inx}) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2}\cos(nx).$$

(b) Does the series converges pointwisely? If yes, prove it. If no, state at which point x the series diverges.

Solution: Yes. For any x, we have

$$\sum_{n=1}^{\infty}\frac{4}{n^2}\cos(nx) \leq \sum_{n=1}^{\infty}\frac{4}{n^2},$$

and the series at the right hand side converges.

(c) Calculate $f(0)$ and use it to find the value of the series $\sum_{n=1}^{\infty}\frac{1}{n^2}$.

Solution: Take $x = 0$, we get

$$f(0) = \pi^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2}, \quad \Rightarrow \quad \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}.$$

(d) Find the value of the series $\sum_{n=1}^{\infty}\frac{1}{n^4}$. *Hint: Apply Plancherel's identity on f.*

Solution: Plancherel's identity says $\|f\|_{L^2}^2 = 2\pi \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2$. We calculate both sides:

$$\|f\|_{L^2}^2 = \int_{-\pi}^{\pi} (\pi - |x|)^4 dx = \frac{2\pi^5}{5},$$

$$2\pi \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 = 2\pi \left(\frac{\pi^4}{9} + 2\sum_{n=1}^{\infty} \frac{4}{n^4} \right) = \frac{2}{9}\pi^5 + 16\pi \sum_{n=1}^{\infty} \frac{1}{n^4}.$$

Therefore, we conclude that $\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{1}{90}\pi^4$.

3. (20 points) Let L_p^2 be the space of 2π-periodic L^2 functions, and H_p^1 be the space of 2π-periodic H^1 functions. Give a *constructive* proof that H_p^1 is dense in L_p^2. Namely, for any given function $f \in L_p^2$, construct a sequence of H_p^1 functions $\{f_n\}$, such that

$$\lim_{n\to\infty} \|f_n - f\|_{L^2} = 0.$$

Note that constructive means f_n should be explicitly defined.

Hint: One way (which is definitely not the only way) to construct the sequence of functions is through its Fourier series.

Solution: Construct f_n as the partial sum of the Fourier series of f.

$$f_n(x) = \sum_{k=-n}^{n} \hat{f}(k) e^{-ikx}.$$

Clearly $f_n \in H_p^1$ (in fact f_n is a trigonometric polynomial which is analytic).

$$\|f_n - f\|_{L^2}^2 = \sum_{k=-\infty}^{\infty} |\hat{f}_n(k) - \hat{f}(k)|^2 = \sum_{|k|>n} |\hat{f}(k)|^2.$$

As $f \in L_p^2$, the series $\sum_{k=-\infty}^{\infty} |\hat{f}(k)|^2$ converges. Therefore, the tail

$$\lim_{n\to\infty} \sum_{|k|>n} |\hat{f}(k)|^2 = 0.$$

This concludes the proof.

4. (20 points) Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a mapping defined as

$$f(x, y) = (e^{-x}, e^x y^3).$$

(a) Prove that f is Fréchet differentiable in $\mathbb{R}^2$. Give an explicit expression of $Df(x)$.

Solution: Compute partial derivatives

$$\partial_x f^1(x,y) = -e^{-x}, \quad \partial_x f^2(x,y) = e^x y^3, \quad \partial_y f^1(x,y) = 0, \quad \partial_y f^2(x,y) = 3y^2 e^x.$$

As all partial derivatives are continuous, the mapping is $\mathcal{C}^1$. In particular, it is Fréchet differentiable. $Df(x)$ is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$. It maps $v \in \mathbb{R}^2$ to Av, where the 2-by-2 matrix A is the Jacobian

$$A(x,y) = \begin{pmatrix} -e^{-x} & 0 \\ e^x y^3 & 3y^2 e^x \end{pmatrix}.$$

(b) Find a point (x, y) where f is locally invertible.

Solution: For example, $(x,y) = (0,1)$, then the Jacobian $A(0,1) = \begin{pmatrix} 1 & 0 \\ 1 & 3 \end{pmatrix}$, which is invertible. So f is locally invertible at $(0,1)$.

(c) Is f globally invertible from $\mathbb{R}^2$ to $f(\mathbb{R}^2)$? If yes, prove it. If no, find two points (x_1, y_1) and (x_2, y_2) that map to the same point.

Solution: Yes. We shall prove f is 1-to-1. Suppose $f(x_1, y_1) = f(x_2, y_2)$. We first have $e^{-x_1} = e^{-x_2}$. As e^{-x} is strictly monotone increasing, we get $x_1 = x_2$. Next, we have $e^{x_1} y_1^3 = e^{x_2} y_2^3$. So $y_1^3 = y_2^3$. This implies $y_1 = y_2$ (even if y^3 has zero derivative at zero). To sum up, f is 1-to-1 and hence f is invertible from $\mathbb{R}^2$ to $f(\mathbb{R}^2)$.

5. (20 points) Ask yourself a question related to the material covered throughout the semester. Explain why it is interesting and nontrivial. Then try to answer it.

Solution: Everyone should have your own solution.