MATH322 Introduction to Mathematical Analysis II **Spring 2016**

Final Exam Name: ____________________

Instructions: *Please read carefully!!*

- Show all work clearly and in order.
- Use no more than **four** consecutive hours of your choice to finish questions 1-4.
- Use as much time as you want for question 5.
- You are allowed to use any resources, but communication with others are not allowed.
- You are required to sign the *honor pledge*.
- Please hand in the exam on Friday April 29 to my office HBH 328, at 10am-12pm, or 3pm-5pm. If the office is closed, slide the exam underneath the door, and send me an email to confirm submission.
- Please attach this page with your name and signature on submission.

Honor Pledge: The Rice University Honor Pledge reads:

> ”On my honor, I have neither given nor received any unauthorized aid on this exam.”

Please write the exact wording of the Pledge, following by your signature, in the space below:

Pledge: ____

____

Your Signature: ____

Gradebox: For grading use. Please leave it blank.

Question:	1	2	3	4	5	Total
Points:	20	20	20	20	20	100
Score:						

Good Luck

1. (20 points) Let f be a positive function defined on $[0, +\infty)$, such that $f \in \mathcal{R}$ on $[0, A]$ for all $A < +\infty$. Moreover, $\lim_{x\to+\infty} f(x) = 1$. The Laplace transform of f, denoted by $\mathcal{L}[f]$, is a function defined on $[0, +\infty)$, with

$$\mathcal{L}[f](\lambda) = \int_0^\infty e^{-\lambda x} f(x)dx.$$

   (a) Prove that $\mathcal{L}[f](\lambda)$ is well-defined for all $\lambda > 0$. Namely, the improper integral converges.

   (b) Prove that $\mathcal{L}[f](0)$ diverges. Moreover,

$$\lim_{\lambda\to 0} \lambda\mathcal{L}[f](\lambda) = 1.$$

2. (20 points) Let f be a 2π-periodic function with $f(x) = (\pi - |x|)^2$ on $[-\pi, \pi]$.

   (a) Find the Fourier coefficients $\hat{f}(n)$ for $n \in \mathbb{Z}$ and write f in terms of Fourier series.

   (b) Does the series converges pointwisely? If yes, prove it. If no, state at which point x the series diverges.

   (c) Calculate $f(0)$ and use it to find the value of the series $\sum_{n=1}^\infty \frac{1}{n^2}$.

   (d) Find the value of the series $\sum_{n=1}^\infty \frac{1}{n^4}$. *Hint: Apply Plancherel's identity on f.*

3. (20 points) Let L_p^2 be the space of 2π-periodic L^2 functions, and H_p^1 be the space of 2π-periodic H^1 functions. Give a *constructive* proof that H_p^1 is dense in L_p^2. Namely, for any given function $f \in L_p^2$, construct a sequence of H_p^1 functions $\{f_n\}$, such that

$$\lim_{n\to\infty} \|f_n - f\|_{L^2} = 0.$$

   Note that constructive means f_n should be explicitly defined.

   Hint: One way (which is definitely not the only way) to construct the sequence of functions is through its Fourier series.

4. (20 points) Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a mapping defined as

$$f(x, y) = (e^{-x}, e^x y^3).$$

   (a) Prove that f is Fréchet differentiable in $\mathbb{R}^2$. Give an explicit expression of $Df(x)$.

   (b) Find a point (x, y) where f is locally invertible.

   (c) Is f globally invertible from $\mathbb{R}^2$ to $f(\mathbb{R}^2)$? If yes, prove it. If no, find two points (x_1, y_1) and (x_2, y_2) that map to the same point.

5. (20 points) Ask yourself a question related to the material covered throughout the semester. Explain why it is interesting and nontrivial. Then try to answer it.