MATH302 Elements of Analysis **Spring 2018**

Homework 8, Due on Tuesday, April 17, 2018

For all the proofs, write *in details* using standard mathematical language.

1. Prove that a set S is dense in $\mathbb{R}$, if and only if

$$\forall\ L \in \mathbb{R}\ \exists\ \{x_n\} \subseteq S\ \ \big(\ (x_n) \text{ converges to } L\ \big).$$

2. Let (x_n) be a convergent sequence.

a). If (x_{n_j}) is a subsequence of (x_n) which converges to L, then (x_n) also converges to L.

b). (Bonus) there exists a monotone subsequence (x_{n_j}) which converges to the same limit.

3. *(Newton's method)* Let $f(x) = x^2 - 2$. Construct a sequence recursively using Newton's method

$$x_1 = 2, \quad x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = \frac{x_n}{2} + \frac{1}{x_n}, \quad \forall\ n \in \mathbb{N}.$$

a). Prove that (x_n) is bounded.

b). Prove that (x_n) is monotone decreasing.

Hint: You can show that x_n is lower bounded by $\sqrt{2}$. Check the following useful identity:

$$x_{n+1} - \sqrt{2} = \frac{(x_n - \sqrt{2})^2}{2x_n}.$$

c). Prove that (x_n) converges, and the limit is a root of $f(x)$.

4. Show that there exists a Cauchy sequence in $\mathbb{Q}$ that does not converge.

5. Ask yourself a question. It could be anything related to this course. State the problem. Try to solve it if you can manage to do it. If not, write down some thoughts about it.