MATH141(0332) Calculus II

Quiz 9, Thursday, November 13, 2008 **Solution of the quiz**

Show all work clearly and in order, and circle your final answers. Justify your answers algebraically whenever possible. Calculator is not allowed in this quiz. You have 25 minutes to take this 12 point quiz. Only 10 points will count. An extra point is offered in Question 4.

1. (*4 points*) Estimate the 5th trancation error for the series $\sum_{n=1}^{\infty} ne^{-n}$.
Hint: You might use integral by parts. You can use $e^{-4} = 0.0183, e^{-5} = 0.0067, e^{-6} = 0.0025$.
Solution:

$$\int_{j+1}^{\infty} f(x)dx \leq E_j \leq \int_{j}^{\infty} f(x)dx,\ where\ f(x) = xe^{-x}$$

First, we should prove that $f(x)$ decreases when $x \geq 5$.

$$f'(x) = e^{-x} + x(-e^{-x}) = (1-x)e^{-x} < 0,\ when\ x \geq 5.$$

So, we have confirmed that we can use the formula to estimate E_5.
Now, we calculate the integral.

$$\int_{j}^{\infty} xe^{-x}dx = \left[-xe^{-x}\right]_j^{\infty} - \int_j^{\infty} -e^{-x}dx = \left[-(x+1)e^{-x}\right]_j^{\infty} = (j+1)e^{-j}$$

Let $j = 5$ and $j = 6$. We have

$$\int_5^{\infty} f(x) = 6e^{-5} = 0.0402, \qquad \int_6^{\infty} f(x) = 7e^{-6} = 0.0175.$$

So, $0.0175 \leq E_5 \leq 0.0402$.

2. (*4 points*) Determine whether the following series converges or diverges.
Hint: You may use comparation rule and then use limit comparation rule.

$$\sum_{n=1}^{\infty} \frac{[sin(n) + cos(5n)]\sqrt[3]{7n^2+5}}{4n^2+2n+3}$$

Solution:

$$a_n = \frac{[sin(n) + cos(5n)]\sqrt[3]{7n^2+5}}{4n^2+2n+3} \Rightarrow |a_n| = \frac{|sin(n) + cos(5n)|\sqrt[3]{7n^2+5}}{4n^2+2n+3}$$

Let

$$b_n = \frac{2\sqrt[3]{7n^2+5}}{4n^2+2n+3}$$

Then as $|sin(n) + cos(5n)| \leq 2$, we know $|a_n| \leq b_n$. So, by comparasion rule, if $\sum b_n$ converges, then $\sum |a_n|$ also converges.
Now consider $\sum b_n$.
Let $c_n = \sqrt[3]{n^2}/n^2 = n^{-4/3}$. By p-test, we know $\sum c_n$ converges.

$$\lim_{n\to\infty} \frac{b_n}{c_n} = \lim_{n\to\infty} \frac{2\sqrt[3]{7n^2+5}}{4n^2+2n+3} \cdot \frac{n^2}{\sqrt[3]{n^2}} = \lim_{n\to\infty} \frac{2\sqrt[3]{7n^2+5}}{\sqrt[3]{n^2}} \cdot \frac{n^2}{4n^2+2n+3} = 2\sqrt[3]{7} \cdot \frac{1}{4} = \frac{\sqrt[3]{7}}{2}$$

So, by limit comparasion rule, we conclude that $\sum b_n$ converges. Then $\sum |a_n|$ converges. So $\sum a_n$ converges, too.

3. (*4 points*) Determine whether the following series converges or diverges.
Hint: You may try to prove that $n!/n^n \leq 1/2$ when $n \geq 2$.

$$\sum_{n=1}^{\infty} \left(\frac{n!}{n^n}\right)^n$$

Solution:
Use root test, the root r should be

$$r = \lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\left(\frac{n!}{n^n}\right)^n} = \lim_{n\to\infty} \frac{n!}{n^n}$$

Now, we consider $n!/n^n$

$$n!/n^n = \frac{\overbrace{n\cdot(n-1)\cdot(n-2)\cdots 2\cdot 1}^{n\ terms}}{\underbrace{n\cdot n\cdot n\cdots n\cdot n}_{n\ terms}} = \overbrace{\underbrace{\frac{n}{n}\cdot\frac{n-1}{n}\cdot\frac{n-2}{n}\cdots\frac{2}{n}}_{n-1\ terms}\cdot\frac{1}{n}}^{n\ terms} < \underbrace{1\cdot 1\cdot 1\cdots 1}_{n-1\ terms}\cdot\frac{1}{n} = \frac{1}{n}$$

So, $n!/n^n < 1/n$.

$$r = \lim_{n\to\infty} \frac{n!}{n^n} \leq \lim_{n\to\infty} \frac{1}{n} = 0$$

As the root is less than 1, we conclude that the series converges.

4. (*1 point*) (Extra point)
Can you find a series which converges but not converges absolutely? Construct the series and briefly prove the statement.
Solution:
There are many series satisfy the condition above. I only give you two examples here.

$$\begin{aligned} a_n &= (-1)^{n+1}\frac{1}{n}, \\ \sum_{n=1}^{\infty} a_n &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = ln(2),\ converges, \\ \sum_{n=1}^{\infty} |a_n| &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots\ diverges \end{aligned}$$

$$\begin{aligned} a_n &= (-1)^{n+1}\frac{1}{2n-1}, \\ \sum_{n=1}^{\infty} a_n &= 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4},\ converges, \\ \sum_{n=1}^{\infty} |a_n| &= 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots\ diverges \end{aligned}$$