MATH141(0332) Calculus II

Quiz 8, November 6 - 11, 2008 **Solution of the quiz**

Show all work clearly and in order, and circle your final answers. Justify your answers algebraically whenever possible. This is a take-home quiz. Please hand in your solution in the discussion on November 11(Tuesday).
This quiz is worth 12 points. 10 of them will be counted to your final score.

1. (*3 points*) Find the Taylor polynomial of the following f for the given value $n = 2$, around the point $x = 0.1$

$$f(x) = \begin{cases} \frac{sin(x)}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$$

Hint: $f'(0) = \lim_{x\to 0} f'(x)$, if the limit exists. To calculate this, you might use L'Hopital's rule.
Solution:

$$f'(0) = \lim_{x\to 0} f'(x) = \lim_{x\to 0} \frac{cos(x)\cdot x - sin(x)}{x^2}$$

Use L'Hopital's rule, we have

$$f'(0) = \lim_{x\to 0} \frac{-sin(x)\cdot x + cos(x) - cos(x)}{2x} = -\frac{1}{2}\lim_{x\to 0} sin(x) = 0$$

Now we calculate $f''(x)$

$$\begin{aligned} f''(x) &= \frac{[-sin(x)\cdot x + cos(x) - cos(x)]x^2 - 2x[cos(x)\cdot x - sin(x)]}{x^4} \\ &= -\frac{sin(x)}{x} - \frac{2[cos(x)\cdot x - sin(x)]}{x^3} \end{aligned}$$

(Notice: you can not just do derivatives on $-\frac{1}{2}sin(x)$ to get $f''(x)$).

$$f''(0) = \lim_{x\to 0} f''(x) = -\lim_{x\to 0}\frac{sin(x)}{x} - \lim_{x\to 0}\frac{2[cos(x)\cdot x - sin(x)]}{x^3}$$

Use L'Hopital's rule on both limits, we have

$$\begin{aligned} f''(0) &= -\lim_{x\to 0}\frac{cos(x)}{1} - \lim_{x\to 0}\frac{2[-sin(x)\cdot x + cos(x) - cos(x)]}{3x^2} \\ &= -1 + \frac{2}{3}\lim_{x\to 0}\frac{sin(x)}{x} \\ &= -1 + \frac{2}{3}\lim_{x\to 0}\frac{cos(x)}{1} \\ &= -1 + \frac{2}{3}\cdot 1 = -\frac{1}{3} \end{aligned}$$

So, from the formula

$$p_2(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 = 1 + \frac{0}{1}\cdot x + \frac{-1/3}{2}\cdot x^2 = 1 - \frac{1}{6}x^2$$

2. (*5 points*) Let $\{a_n\}_{n=1}^{\infty}$ be the sequence $\sqrt{6}$, $\sqrt{6+\sqrt{6}}$, $\sqrt{6+\sqrt{6+\sqrt{6}}}$, ..., where in general, $a_{n+1} = \sqrt{6+a_n}$.

(1) Show that for every positie integer n, $0 < a_n < 3$. (Notice:$\sqrt{6} < 3$)
Solution:
Use induction.
i) When $n = 1$, $a_1 = \sqrt{6} \in (0, 3)$.
ii) Assume $0 < a_n < 3$. Then

$$a_{n+1} = \sqrt{6+a_n} > \sqrt{6+0} = \sqrt{6} > 0$$

$$a_{n+1} = \sqrt{6+a_n} < \sqrt{6+3} = \sqrt{9} = 3$$

So we have got $0 < a_{n+1} < 3$.
From i) and ii), we conclude that $0 < a_n < 3$ for every n.

(2) Show that for every positie integer n, $a_n \leq a_{n+1}$.
Hint: $a_{n+1} - a_n = a_{n+1} - (a_{n+1}^2 - 6)$. You might find the result of (1) is useful to prove (2).
Solution:
As $a_{n+1} = \sqrt{6+a_n}$, we have $a_n = a_{n-1}^2 - 6$. So,

$$a_{n+1} - a_n = a_{n+1} - (a_{n+1}^2 - 6) = -a_{n+1}^2 + a_{n+1} + 6) = -(a_{n+1} - 3)(a_{n+1} + 2).$$

From (1), we know that $0 < a_{n+1} < 3$ for all n. So$-(a_{n+1} - 3)(a_{n+1} + 2) > 0$. Then we have $a_{n+1} - a_n > 0$. So $a_n < a_{n+1}$.

(3) Show that $\{a_n\}$ converges, and find the value of $\lim_{n\to\infty} a_n$.
Solution:
From (1)(2), we know that $\{a_n\}$ is a bounded increasing sequence, so $\{a_n\}$ converges.
Let $L = \lim_{n\to\infty} a_n$. We know for every n, $a_{n+1} = \sqrt{6+a_n}$. So

$$L = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{6+a_n} = \sqrt{6 + \lim_{n\to\infty} a_n} = \sqrt{6+L}$$

So, $L^2 - L - 6 = 0$. Then $L = 3$. (Another root $L = -2$ can not be a limit of a positive sequence)
In another word, the value of $\lim_{n\to\infty} a_n$ is 3.

3. (*4 points*) Determine whether the following infinite sum converges or diverges. If it converges, find its sum. (Use the back side of the paper to finish this problem.)

$$(1)\ \sum_{n=1}^{\infty}(-1)^{n+1}\frac{2^{2n+1}}{5^{n-2}} \qquad (2)\ \sum_{n=1}^{\infty}\frac{3n^3-n}{\sqrt{n+1}}$$

Solution for (1):

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2^{2n+1}}{5^{n-2}} = \sum_{n=1}^{\infty}(-1)\cdot(-1)^n\frac{2\cdot 2^{2n}}{\frac{1}{25}\cdot 5^n} = \sum_{n=1}^{\infty}(-50)\frac{(-1)^n\cdot 4^n}{5^n} = \sum_{n=1}^{\infty}(-50)\left(-\frac{4}{5}\right)^n$$

So, $r = -\frac{4}{5}$, $|r| < 1$. The series converges.

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2^{2n+1}}{5^{n-2}} = \sum_{n=1}^{\infty}(-50)\left(-\frac{4}{5}\right)^n = \frac{(-50)\cdot(-\frac{4}{5})^1}{1-(-\frac{4}{5})} = \frac{200}{9}$$

Solution for (2):

$a_n = \frac{3n^3-n}{\sqrt{n+1}}$. Let $b_n = \frac{3n^3}{\sqrt{n}} = 3n^{\frac{5}{2}}$. Then we have

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\frac{3n^3-n}{\sqrt{n+1}}}{\frac{3n^3}{\sqrt{n}}} = \lim_{n\to\infty}\frac{3n^3-n}{3n^3}\cdot\frac{\sqrt{n}}{\sqrt{n+1}} = \lim_{n\to\infty}(1-\frac{1}{3n^2})\cdot\frac{1}{\sqrt{1+\frac{1}{n}}} = 1$$

So we can use limit comparasion rule on $\{a_n\}$ and $\{b_n\}$.
Since $\sum_{n=1}^{\infty} b_n$ diverges based on p-test, $\sum_{n=1}^{\infty} a_n$ also diverges.

Alternative solution for (2):

$a_n = \frac{3n^3-n}{\sqrt{n+1}}$. So

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{3n^3-n}{\sqrt{n+1}} = +\infty$$

So $\lim\limits_{n\to\infty} a_n \neq 0$. Then the series diverges.