STAT100 Elementary Statistics and Probability

Exam 2, Sample Test, Summer 2014 **Solution**

Show all work clearly and in order, and circle your final answers. Justify your answers algebraically whenever possible. Calculators are allowed in this exam for basic calculation only. You have 80 minutes to take this 104 point exam. If you get more than 100 points, your grade will be 100.

1. (24 points)

   (a) (12 points) Last year, 75% of UMD students read the Diamondback. In a group of 1200 randomly selected students, what is the approximate probability that fewer than or equal to 880 students read the Diamondback?

   Solution: Let X be the number of students out of 1200 read the Diamondback. Clearly, X has the binomial distribution Binom$(1200, .75)$. **(2 points)**
   The goal is to find $P(X \leq 400)$. Approximately, X has a distribution close to $N(np, npq) = N(900, 225)$. **(6 points)**
   Using the .5 correction rule, we compute
   $P(X \leq 400.5) = P\left(\frac{X-900}{15} \leq \frac{880.5-900}{15}\right) = \Phi\left(\frac{880.5-900}{15}\right) = \Phi(-1.3).$
   (9 points) Look up the normal table, we get $\Phi(-1.3) = .0968$. **(12 points)**

   (b) (12 points) To determine if this percentage has changed, a random sample of 300 students is studied and 246 read the Diamondback. Does this finding indicate that the current percentage of students reading the Diamondback has changed from what is was last year? (Form the testing hypotheses, compute the $P-$value and determine if the percentage has changed or not based on the data.)

   Solution: Let p denote the current percentage of students reading the Diamondback. Form the test

   $$H_0: \ p = .75 \quad \text{vesus} \quad H_1: \ p \neq .75. \ \textbf{(3 points)}$$

   The test statistic is

   $$Z = \frac{\hat{p} - .75}{\sqrt{.75 \times .25/300}}. \ \textbf{(5 points)}$$

   Compute the value of Z from the sample data

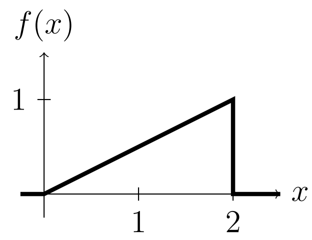
   $$Z = \frac{(246/300) - .75}{\sqrt{.75 \times .25/300}} = 2.8. \ \textbf{(8 points)}$$

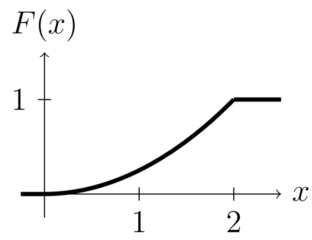
The $P-$value corresponds to the data is

$$P\text{-value} = P(|Z| \geq 2.8) = 2\Phi(-2.8) = .0052. \quad \textbf{(11 points)}$$

As the $P-$value is very small, the evidence against H_0 is strong. It indicates that the percentage has changed. **(12 points)**

2. (20 points) A continuous random variable X has the following probability density function f and cumulative distribution function F.





$$f(x) = \begin{cases} x/2 & 0 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases} \qquad F(x) = \begin{cases} 0 & x \leq 0 \\ x^2/4 & 1 \leq x \leq 2 \\ 1 & x \geq 2 \end{cases}$$

(a) (5 points) Which of the two intervals $[0 < X < .5]$ or $[1.5 < X < 2]$ is assigned a higher probability?

Solution: $[1.5 < X < 2]$ has higher probability. **(5 points)**

(b) (5 points) Compute $P(.5 \leq X \leq 1.5)$.

Solution: Using pdf: The corresponding region is a trapezoid. Its area reads $(.25 + .75) \cdot 1/2 = .5$. **(5 points)**
Using cdf: $P(.5 \leq X \leq 1.5) = F(1.5) - F(.5) = .5625 - .0625 = .5$. **(5 points)**

(c) (5 points) Determine the median of X.

Solution: Using cdf: the median m satisfies $F(m) = .5$. Therefore, $m^2/4 = .5$. It yields $m = \sqrt{2} \approx 1.414$. **(5 points)**

(d) (5 points) Is X left-skewed, right-skewed or symmetric?

Solution: From pdf, we know X is left-skewed. **(5 points)**

3. (20 points)

(a) (5 points) Φ is the cumulative distribution function of standard normal distribution. Find $\Phi(1.222)$ and $\Phi(1.342)$. Use the normal table and linear interpolation to get your answer. (Round up to 4 digits)

Solution: Look up the normal table and get $\Phi(1.22) = .8888, \Phi(1.23) = .8907$.

Interpolate $\Phi(1.222) = \dfrac{.8907(1.222 - 1.22) + .8888(1.23 - 1.222)}{1.23 - 1.22} = .8892.$

Similarly, look up the normal table and get $\Phi(1.34) = .9099, \Phi(1.35) = .9115$.

Interpolate $\Phi(1.342) = \dfrac{.9115(1.342 - 1.34) + .9099(1.35 - 1.342)}{1.35 - 1.34} = .9102.$

(b) (5 points) Suppose Z is a random variable with standard normal distribution. Find $P(|Z| \leq 1.222)$.

Solution:

$$P(|Z| \leq 1.222) = 2\Phi(1.222) - 1 = .7784.$$

(c) (5 points) Suppose X has a normal distribution with mean 4 and standard deviation 2. Find $P(6.444 \leq X \leq 6.684)$.

Solution:

$$\begin{aligned} P(6.444 \leq X \leq 6.684) =& P\left(\frac{6.444 - 4}{2} < \frac{Y - 4}{2} < \frac{6.684 - 4}{2}\right) \\ =& \Phi(1.342) - \Phi(1.222) = .021 \end{aligned}$$

(d) (5 points) [*] Suppose Y has a normal distribution with mean -6.41 and standard deviation 5. Find $P(|Y| > .3)$.

Solution:

$$\begin{aligned} P(|Y| > .3) =& P(Y < -.3) + P(Y > .3) \\ =& P\left(\frac{Y + 6.41}{5} < \frac{-.3 + 6.41}{5}\right) + P\left(\frac{Y + 6.41}{5} > \frac{.3 + 6.41}{5}\right) \\ =& \Phi(1.222) + (1 - \Phi(1.342)) = .979. \end{aligned}$$

4. (20 points) The number of complaints per day, X, received ny a cable TV distributor has the probability distribution

x	0	1	2	3
$f(x)$	.4	.3	.1	.2

(a) (6 points) Find the expectation and standard deviation of the number of complaints per day.

Solution: (3 points for expectation and 3 points for standard deviation)

	x_i	$f(x_i)$	$x_i f(x_i)$	x_i^2	$x_i^2 f(x_i)$
	0	.4	0	0	0
	1	.3	.3	1	.3
	2	.1	.2	4	.4
	3	.2	.6	9	1.8
$\sum$		1	$\mu = 1.1$		$\mathbb{E}X^2 = 2.5$

So, $\mathbb{E}X = 1.1$, $\text{Var}X = \mathbb{E}X^2 - (\mathbb{E}X)^2 = 1.29$ and $\text{Sd}X = \sqrt{1.29} = 1.136$.

(b) (6 points) Find the expectation and standard deviation of the average number of complaints per day in 90 days. *(i.e., find $\mathbb{E}\bar{X}$ and Sd($\bar{X}$), where $\bar{X} = (X_1 + \cdots + X_{90})/90$.)*

Solution: $\mathbb{E}\bar{X}_1 = \mathbb{E}X_1 = 1.1$. **(3 points)**
$\text{Var}\bar{X} = \frac{1}{n}\text{Var}X_1 = 1.29/90 = .0143$. So $\text{Sd}\bar{X} = \sqrt{.0143} = .1197$. **(6 points)**

(c) (8 points) What is the approximate probability that the distributor will receive more that 125 complaints in 90 days.

Solution: Note that if the number of complaints is more than 125, the sample mean will be greater than 125/90. The goal is to compute $P(\bar{X} > 125/90)$. **(3 points)**
Since the simple size $n = 90$ is large, from the central limit theorem, we approximate the distribution of $\bar{X}$ by a normal distribution with mean 1.1 and standard deviation .1197. **(5 points)**
Indeed, we have $P(\bar{X} > 125/90) = P\left(\frac{X - 1.1}{.1197} > \frac{(125/90) - 1.1}{.1197}\right) = 1 - \Phi(2.41) = .0080$. **(8 points)**

5. (20 points) A credit company randomly selected 60 contested items and recorded the dollar amount being contested. These contested items had sample mean $\bar{x} = 75$ dollars and sample variance $s^2 = 9$.

(a) (4 points) Give a point estimate of the population mean contested amount μ.

Solution: The sample mean 75 is a point estimate of the population mean. **(4 points)**

(b) (6 points) Calculate a 90% confidence interval for the population mean contested amount.

Solution: We have $n = 60, \bar{x} = 75, s = 3, \alpha = .1$. Look up the normal table, $z_{\alpha/2} = z_{.05} = 1.645$. **(2 points)**
Using the formula, the 90% confidence interval is given by

$$\left[75 - 1.645\frac{3}{\sqrt{60}}, 75 + 1.645\frac{3}{\sqrt{60}}\right] = [74.363, 75.637]. \quad \textbf{(6 points)}$$

(c) (10 points) We formulate the following hypotheses

$$H_0: \ \mu = 74 \quad \text{versus} \quad H_1 : \mu > 74.$$

Take $\alpha = .008$ (meaning that the probability of making type I error is less than .8%). What does the data suggest, rejecting H_0 or retaining H_0?

Solution: The test statistic is $Z = \dfrac{\bar{X} - 74}{3/\sqrt{60}}$. **(2 points)**
We shall reject H_0 if $Z \geq z_\alpha$, and retain H_0 if $Z < z_\alpha$. **(4 points)**
Compute $z_\alpha = z_{.008} = 2.41$, **(7 points)** and $Z = \dfrac{75 - 74}{3/\sqrt{60}} = 2.582$. **(10 points)**
As $2.582 > 2.41$, the data suggest us to reject H_0. **(12 points)**