

STAT100 Elementary Statistics and Probability

Exam 2, Monday, August 11, 2014

Solution

Show all work clearly and in order, and circle your final answers. Justify your answers algebraically whenever possible. Calculators are allowed in this exam for basic calculation only. You have 80 minutes to take this 104 point exam. If you get more than 100 points, your grade will be 100.

1. (18 points) Suppose the weights of the contents of cans of mixed nuts have a normal distribution with mean 32.4 ounces and a standard deviation of .4 ounce.
- (a) (6 points) If every can is labeled 32 ounces, what proportion of the cans have contents that weight less than the labeled amount?

Solution: Let X be a random variable representing the weight of a can. Based on the assumption, $X \sim \mathcal{N}(32.4, .16)$. **(2 points)**

The event that a can has weight less than 32 amount can be expressed by $X < 32$. **(3 points)** We find probability of the event:

$$P(X < 32) = P\left(\frac{X - 32.4}{.4} < \frac{32 - 32.4}{.4}\right) = P(Z < -1) = \Phi(-1) = .1587.$$

Therefore, 15.87% cans have weight less than 32 ounce. **(6 points)**

- (b) (6 points) If 10 packages are randomly selected, what is the probability that the average weight is less than 32 ounces?

Solution: The average weight of 10 packages \bar{X} has a normal distribution, $\bar{X} \sim \mathcal{N}(32.4, .016)$. **(2 points)**

The event that a can has weight less than 32 amount can be expressed by $\bar{X} < 32$. **(3 points)** We find probability of the event:

$$P(\bar{X} < 32) = P\left(\frac{\bar{X} - 32.4}{\sqrt{.016}} < \frac{32 - 32.4}{\sqrt{.016}}\right) = P(Z < -3.16) = \Phi(-3.16) = .0008.$$

(6 points)

- (c) (6 points) How many packages has to be randomly selected to be 90% sure that the error margin (between the average weight and 32.4 ounce) does not exceed .1?

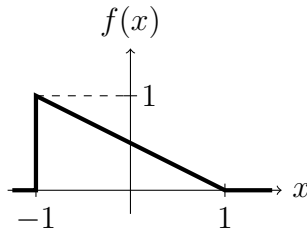
Solution: We have $\sigma = .4, d = .1, \alpha = .1$ and hence $z_{\alpha/2} = z_{.05} = \Phi^{-1}(.95) =$

1.645. (4 points) Using the formula,

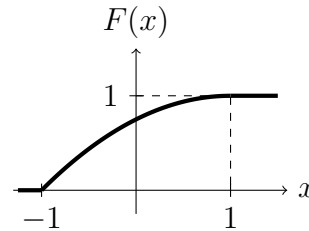
$$n \geq \left(\frac{z_{\alpha/2}\sigma}{d} \right)^2 = \left(\frac{1.645 \cdot .4}{.1} \right)^2 \approx 43.296.$$

So, at least 44 packages has to be selected. (6 points)

2. (20 points) A continuous random variable X has the following probability density function f and cumulative distribution function F .



$$f(x) = \begin{cases} \frac{1-x}{2} & -1 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$



$$F(x) = \begin{cases} 0 & x \leq -1 \\ \frac{-x^2 + 2x + 3}{4} & -1 \leq x \leq 1 \\ 1 & x \geq 1 \end{cases}$$

- (a) (5 points) Which of the two intervals $[-1 < X < 0]$ or $[0 < X < 1]$ is assigned a higher probability?

Solution: $[-1 < X < 0]$ has higher probability. (5 points)

- (b) (5 points) Compute $P(|X| \leq .4)$.

Solution: Using pdf: The corresponding region is a trapezoid. Its area reads $(.7 + .3) \cdot .8/2 = .4$. (5 points)

Using cdf: $P(-.4 \leq X \leq .4) = F(.4) - F(-.4) = .99 - .59 = .4$. (5 points)

- (c) (5 points) Determine the third quartile of X .

Solution: Using cdf: the third quartile Q_3 satisfies $F(Q_3) = .75$. (2 points)
Therefore,

$$\frac{-Q_3^2 + 2Q_3 + 3}{4} = .75 \Rightarrow Q_3^2 - 2Q_3 = 0 \Rightarrow Q_3 = 0 \text{ or } Q_3 = 2.$$

As Q_3 is between -1 and 1, we conclude $Q_3 = 0$. (5 points)

- (d) (5 points) Is X left-skewed, right-skewed or symmetric?

Solution: From pdf, we know X is right-skewed. (5 points)

3. (24 points)

- (a) (6 points) Φ is the cumulative distribution function of standard normal distribution. Find $\Phi(.587)$ and $\Phi(1.032)$. Use the normal table and linear interpolation to get your answer. (Round up to 4 digits)

Solution: Look up the normal table and get $\Phi(.58) = .7190$, $\Phi(.59) = .7224$.

Interpolate $\Phi(.587) = .7190 + \frac{.587 - .58}{.59 - .58}(.7224 - .7190) \approx .7214$.

Similarly, look up the normal table and get $\Phi(1.03) = .8485$, $\Phi(1.04) = .8508$.

Interpolate $\Phi(1.032) = .8485 + \frac{.1.032 - 1.03}{1.04 - 1.03}(.8508 - .8485) \approx .8490$.

- (b) (5 points) Suppose Z is a random variable with standard normal distribution. Find $P(Z \leq .587 \text{ or } Z \geq 1.032)$.

Solution: $P(|Z| \leq 1.222) = \Phi(.587) + (1 - \Phi(1.032)) = .8724$.

- (c) (6 points) Suppose X has a normal distribution with mean -1.619 and standard deviation 2 . Find $P(|X| > .445)$.

Solution:

$$\begin{aligned} P(|X| > .445) &= P(X > .445) + P(X < -.445) \\ &= P\left(\frac{X + 1.619}{2} < \frac{.445 + 1.619}{2}\right) + P\left(\frac{X + 1.619}{2} > \frac{-.445 + 1.619}{2}\right) \\ &= P(Z < .587) + P(Z > 1.032) = .8724. \end{aligned}$$

- (d) (7 points) [*] Find the 70th quartile of standard normal distribution with cumulative distribution function Φ . Use the normal table and linear interpolation to get your answer. (Round up to 4 digits)

Solution: From the standard normal table, we find two z values where $\Phi(z)$ are closest to $.7$: $\Phi^{-1}(.6985) = .52$ and $\Phi(.7019) = .53$.

Interpolate $\Phi^{-1}(.7) = .52 + \frac{.7 - .6985}{.7019 - .6985}(.53 - .52) \approx .5244$.

(4 points if the answer is $.52$, with no linear interpolation.)

4. (20 points)

- (a) (10 points) Last year, 75% of UMD students read the Diamondback. In a group of 1200 randomly selected students, what is the approximate probability that fewer than or equal to 880 students read the Diamondback?

Solution: Let X be the number of students out of 1200 read the Diamondback. Clearly, X has the binomial distribution $\text{Binom}(1200, .75)$. **(2 points)**

The goal is to find $P(X \leq 880)$. Approximately, X has a distribution close to $N(np, npq) = N(900, 225)$. **(5 points)**

Using the .5 correction rule, we compute

$$P(X \leq 880.5) = P\left(\frac{X - 900}{15} \leq \frac{880.5 - 900}{15}\right) = \Phi\left(\frac{880.5 - 900}{15}\right) = \Phi(-1.3).$$

(8 points) Look up the normal table, we get $\Phi(-1.3) = .0968$. **(10 points)**

- (b) (10 points) To determine if this percentage has changed, a random sample of 300 students is studied and 246 read the Diamondback. Does this finding indicate that the current percentage of students reading the Diamondback has changed from what it was last year? (Form the testing hypotheses, compute the P -value and determine if the percentage has changed or not based on the data.)

Solution: Let p denote the current percentage of students reading the Diamondback. Form the test

$$H_0 : p = .75 \quad \text{versus} \quad H_1 : p \neq .75. \quad \text{(2 points)}$$

The test statistic is

$$Z = \frac{\hat{p} - .75}{\sqrt{.75 \times .25/300}}. \quad \text{(4 points)}$$

Compute the value of Z from the sample data

$$Z = \frac{(246/300) - .75}{\sqrt{.75 \times .25/300}} = 2.8. \quad \text{(6 points)}$$

The P -value corresponds to the data is

$$P\text{-value} = P(|Z| \geq 2.8) = 2\Phi(-2.8) = .0052. \quad \text{(8 points)}$$

As the P -value is very small, the evidence against H_0 is strong. It indicates that the percentage has changed. **(10 points)**

5. (22 points) A manager at a power company monitored the employee time required to process high-efficiency lamp bulb rebates. A random sample of 40 applicants gave a sample mean time of 3.8 minutes and a standard deviation of 1.2 minutes.

- (a) (8 points) Calculate a 95% confidence interval for the population mean time μ .

Solution: We have $n = 40, \bar{x} = 3.8, s = 1.2, \alpha = .05$. Look up the normal table, $z_{\alpha/2} = z_{.025} = \Phi^{-1}(.975) = 1.96$. **(4 points)**

Using the formula, the 90% confidence interval is given by

$$\left[3.8 - 1.96 \frac{1.2}{\sqrt{40}}, 3.8 + 1.96 \frac{1.2}{\sqrt{40}} \right] = [3.428, 4.172]. \quad \text{(8 points)}$$

- (b) (10 points) Is the claim that $\mu > 3.5$ minutes substantiated by these data? Test with level of significance $\alpha = .1$.

Solution: We formulate the test hypothesis:

$$H_0 : \mu = 3.5, \quad H_1 : \mu > 3.5. \quad \text{(2 points)}$$

The test statistic is $Z = \frac{\bar{X} - 3.5}{1.2/\sqrt{40}}$. **(4 points)**

We shall reject H_0 if $Z \geq z_\alpha$, and retain H_0 if $Z < z_\alpha$. Compute $Z = \frac{3.8 - 3.5}{1.2/\sqrt{40}} = 1.581$ **(6 points)** and $z_\alpha = z_{.1} = \Phi^{-1}(.9) = 1.28$, **(8 points)**

As $1.581 > 1.28$, the data suggest us to reject H_0 , namely, it substantiate the claim that $\mu > 3.5$. **(10 points)**

- (c) (4 points) Based on your decision in Part (b), what error could you have possibly made? Explain in the context of the problem.

Solution: We could make type-I error. In the context of the problem, type-I error means: the actual time required to process high-efficiency lamp bulb rebates is 3.5 minutes, but we claim that the time is greater than 3.5 minutes. **(4 points)**